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Pion-Pion Interaction and Multiple Pion Production in Nucleon-Antinucleon Annihilation Process.

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Summary. — The multiple pion production in antinucleon-nucleon annihilation processes is investigated by means of the statistical theory with the strong pion-pion interaction. The characteristics of the isobar state of two pions are determined from the observed momentum distribution of a nucleon in the reaction $\pi^- + p \to N + 2\pi$ at 1 GeV or from the theories of Takeda, Dyson and Miyazawa. The mean number of produced pions is $3 \div 4$ and rather less than that given by experiment.

1. – Introduction.

According to the statistical theory proposed originally by FERMI [1], we can understand qualitatively the phenomenon of the multiple production of pions in high energy nucleon-nucleon and pion-nucleon collisions. Considering the effect of the interactions of final particles, we obtain more satisfactory agreements with experiments in GeV regions. Investigations for the effects of the strong interactions of the particles in the final states are given by many authors [2] and one of the most elegant methods that included the effects of final interactions in Fermi's statistical theory was proposed by BELEN'KIJ [3].

In this note, we will apply the statistical theory to the phenomena of the

[1] E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950).

[2] D. C. PEASLEE: *Phys. Rev.*, **94**, 1085 (1954); **95**, 1580 (1954); S. T. LINDENBAUM and R. M. STERNHEIMER: *Phys. Rev.*, **105**, 1874 (1957); S. BARSHAY: *Phys. Rev.*, **106**, 572 (1957).

[3] S. Z. BELEN'KIJ: *Nucl. Phys.*, **2**, 259 (1956).

multiple production of pions in nucleon-antinucleon annihilation process. Experiments ([4]) carried out recently show that though the energy and angular distribution of the produced pions are not inconsistent with the statistical theory, the observed multiplicity of pions is larger than that predicted by Fermi's theory. To get the large multiplicity, we will consider the effect of the interactions of particles in the final states. In the case of the nucleon-antinucleon annihilation process, the interaction in the final states means the strong pion-pion interaction, since no nucleon is present in the final state.

The presence of the strong pion-pion interactions was assumed by many authors, even though no direct and decisive proof for it has been obtained experimentally. TAKEDA ([5]) and DYSON ([6]) proposed the « strong pion-pion interaction model » to interprete the second maximum of the total cross-section of the pion-nucleon scattering. In his analysis of extreme high energy phenomena, KOBA ([7]) also concluded that the modification of Fermi's original theory in favour of Landau's ([8]) hydrodynamical theory implied the existence of a strong pion-pion interaction at a lower energy, say ~ 1 GeV. From the strong coupling treatment of the meson theory, MIYAZAWA ([9]) predicted the presence of isobar states of two pions. From the reaction « $\pi^- + p \rightarrow$ a nucleon of two pions » at 1 GeV observed by WALKER *et al.* ([10]), we may conclude that the momentum distribution of a final nucleon or a relative kinetic energy distribution of two pions is not inconsistent with the presence of an isobar state of two pions in accordance with Belen'kij's statistical theory. (See next section).

In contrast with the treatment given by SUDARSHAN ([11]) which was based on Pomerančuk's model, we will take the isobar model of two pions which follows from the assumption of the presence of a resonant state of two pions and the statistical theory modified by BELEN'KIJ.

2. – Information from the reaction « $\pi^- + p \rightarrow N + 2\pi$ » at 1 GeV and multiplicity of the produced pions.

From the experimental results obtained by WALKER *et al.*, we see that the momentum distribution of a nucleon in the reaction « $\pi^- + p \rightarrow N + 2\pi$ »

([4]) W. H. BARKAS, R. W. BIRGE, W. W. CHUPP, A. G. EKSPONG, G. GOLDHABER, S. GOLDHABER, H. H. HECKMAN, D. H. PERKINS, Y. SANDWEISS, E. SEGRÈ, F. M. SMITH, D. H. STORK and L. VAN ROSSUM: *Phys. Rev.*, **105**, 1037 (1957); *VII Rochester Conference*.

([5]) G. TAKEDA: *Phys. Rev.*, **100**, 440 (1955).

([6]) F. DYSON: *Phys. Rev.*, **99**, 1037 (1955).

([7]) Z. KOBA: *Progr. Theor. Phys.*, **15**, 461 (1956).

([8]) S. Z. BELEN'KIJ and L. D. LANDAU: *Usp. Fiz. Nauk*, **56**, 309 (1955).

([9]) H. MIYAZAWA: *Phys. Rev.*, **97**, 1399 (1955).

([10]) W. D. WALKER, F. HUSHFAR and W. D. SHEPHARD: *Phys. Rev.*, **104**, 526 (1956).

([11]) G. SUDARSHAN: *Phys. Rev.*, **103**, 777 (1956).

deviates from that given by Fermi's original statistical theory. (See Fig. 1.) It is easy to understand this fact by means of Belen'kij's modification of the statistical theory with the assumption of the presence of a resonant state of two pions. According to Belen'kij, in the case of the reaction « $\pi + \mathrm{N} \to \mathrm{N} + 2\pi$ » the statistical weight is given as follows,

$$S(\mathrm{N}, 2\pi) = S_0(\mathrm{N}, 2\pi) + \sum_l (2l+1) \int S_0(\mathrm{N}, 1_{\pi\pi}(w')) \frac{\partial \eta_l(w')}{\partial w'} \mathrm{d}w' \,, \tag{1}$$

where $S_0(\mathrm{N}, 2\pi)$ is the statistical weight of a nucleon and two pions given by Fermi's theory and $\eta_l(w')$ is the phase shift of pion-pion scattering in a given angular momentum l at energy w'. $S_0(\mathrm{N}, 1_{\pi\pi}(w'))$ is the statistical weight of a nucleon and a virtual particle of mass W' given by Fermi's theory. If the resonant state is present and all other contributions are neglected, Eq. (1) becomes

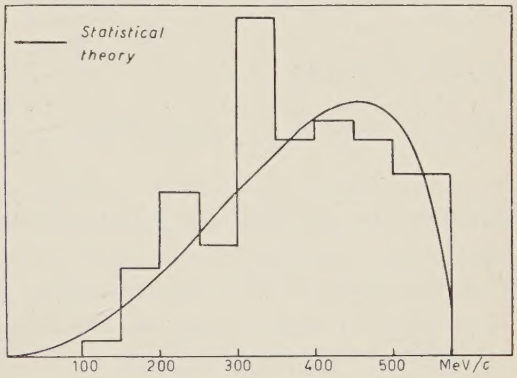

Fig. 1.

$$S(\mathrm{N}, 2\pi) = S_0(\mathrm{N}, 2\pi) + (2l_0 + 1)\, S_0(\mathrm{N}, 1_{\pi\pi}(w_0)) \tag{2}$$

and hence the momentum distribution of a final nucleon is as follows,

$$D(P_\mathrm{N}) = D_0(P_\mathrm{N}) + (2l_0 + 1)\ \mathrm{const}\ \delta(P_\mathrm{N} - P_0) \,. \tag{3}$$

The first term in Eq. (3) is the ordinary statistical distribution and the second term represents the effect of the pion-pion interaction. Though, of course, the pion nucleon interaction in the final states is important in general, its effect is smeared out and may not be remarkable in the momentum distribution of a final nucleon. Perhaps it is remarkable in the momentum distribution of a final pion, especially of a positive pion in the reaction $\pi^- + \mathrm{p} \to \pi^+ + \pi^- + \mathrm{n}$, since there is a resonant state of a pion and a nucleon system having $I = \frac{3}{2}$ and $J = \frac{3}{2}$.

From Eq. (3) and Fig. 1, we see that the resonance energy of the pion-pion interaction is equal to $Q_{\pi\pi}$ (relative kinetic energy between two pions) corresponding to the nucleon momentum ~ 350 MeV/c where the momentum distribution deviates appreciably from that given by Fermi's theory. This value of $Q_{\pi\pi}$ is equal to $(350 \div 400)$ MeV. Isotopic spin I and orbital angular momentum of a two pion system is in general ($I = 0$ or 2, $l = 2n$) or ($I = 1$, $l = 2n+1$) where n is an integer.

Thus, under the assumption that the energy of the isobar state including the rest energies of two pions is ~ 670 MeV and the isotopic spin and the angular momentum of this state is $(I=0,\ l=0)$ or $(I=1,\ l=1)$ respectively, we calculated the multiplicity of pions in antinucleon-nucleon annihilation process according to Belen'kij's theory. For the evaluation of the phase space integral, we took the K-mesons and the unstable two pion systems in the isobar state to be non-relativistic particles and pions to be extreme relativistic particles. In this approximation, the probability for various processes is as follows:

for $\mathrm{N}+\overline{\mathrm{N}} \to \mathrm{n}_\pi\pi$:

$$P(n_\pi) = \text{const}\ S_{n_\pi} T_{n_\pi} \cdot (\Omega/(2\pi\hbar)^3)^{n_\pi - 1} \cdot \frac{(4n_\pi - 4)}{(2n_\pi - 1)!(2n_\pi - 2)!(2n_\pi - 4)!}\, W^{3(n_\pi - 1)-1}$$

for $\mathrm{N}+\overline{\mathrm{N}} \to \mathrm{n}_\pi\pi + \mathrm{n}_\mathrm{K}(\mathrm{K}\ \text{or}\ \overline{\mathrm{K}}) + \mathrm{n}_{\pi\pi}(\pi\text{-}\pi)$:

$$P(n_\pi, n_\mathrm{K}, n_{\pi\pi}) = \text{const}\ S_{n_\pi n_\mathrm{K} n_{\pi\pi}} \cdot T_{n_\pi n_\mathrm{K} n_{\pi\pi}} \cdot (\Omega/(2\pi\hbar)^3)^{n_\pi + n_\mathrm{K} + n_{\pi\pi} - 1} \cdot$$
$$\cdot (8\pi^3)^{n_\mathrm{K} + n_{\pi\pi}} \frac{(m_\mathrm{K}^{n_\mathrm{K}} + m_{\pi\pi}^{n_{\pi\pi}})^{\frac{3}{2}}}{(n_\mathrm{K} m_\mathrm{K} + n_{\pi\pi} m_{\pi\pi})^{\frac{3}{2}}} \cdot \frac{(\overline{w} - n_\mathrm{K} m_\mathrm{K} - n_{\pi\pi} m_{\pi\pi} - n_\pi m_\pi)^{\frac{3}{2}(n_{\pi\pi} + n_\mathrm{K}) + 3n_\pi - \frac{5}{2}}}{\Gamma(3(n_\mathrm{K} + n_{\pi\pi} - 1)/2 + 3n_\pi)},$$

$n_\pi n_\mathrm{K} n_{\pi\pi}$: numbers of pions, K-mesons and unstable two pion states respectively;

$m_\pi m_\mathrm{K} m_{\pi\pi}$: a rest energy of a pion, a K-meson and an unstable two pion state respectively;

T and S: statistical weight from the isotopic spin and the spin;

Ω: the interaction volume (taken to be $(4\pi/3)(\hbar/m_\pi c)^3 \equiv \Omega_0$);

$W = 2m_\mathrm{N}c^2$ (we considered the annihilation at rest).

For the K-meson production, according to NISHIJIMA and GELL-MANN we assume $I_\mathrm{K} = \frac{1}{2}$ and $S_\mathrm{K}\,(\text{spin}) = 0$ and the conservation of strangeness.

In Table I, we summarize the number distribution of produced pions in the case of $(I_{\pi\pi}=0,\ l_{\pi\pi}=0)$ and $(I_{\pi\pi}=1,\ l_{\pi\pi}=1)$. For comparison, the re-

TABLE I.

Case (I, l) \ n_π		0	1	2	3	4	5	6
$\bar{\mathrm{p}}\mathrm{p}$	(0, 0)	3	20	8	20	38	12	1
	(1, 1)	2	8	3	20	50	18	1
$\bar{\mathrm{n}}$	(0, 0)	2	19	7	26	29	18	1
	(1, 1)	1	8	3	16	53	19	0
FERMI		6	27	12	38	15	3	0

sults given by Fermi's theory without any final interactions are also shown. In Table II, the mean number of pions and the probability of K-meson production are tabulated.

TABLE II.

(I, l)		n_π	p_K
(0, 0)	pn	3.18	$\sim 24\%$
	pp	3.09	
(1, 1)	pn	3.69	$\sim 14\%$
	np	3.63	
FERMI		2.3	$\sim 40\%$

Although we obtain the larger multiplicity of pions by virtue of the statistical theory including the strong pion-pion interaction than that given Fermi's original theory which does not include the effect of any final state interaction, it is still much less than that given by experiments. In order to obtain a good agreement with the observed multiplicity of pions, we have to take the interaction volume $\Omega \cong 10\Omega_0$ (where $\Omega_0 = (4\pi/3)(\hbar/m_\pi c)^3$). The results are shown in Table III for different interaction volumes in the case of $(I=1, l=1)$.

TABLE III.

	Ω_0	$3\Omega_0$	$5\Omega_0$	$10\Omega_0$
$\bar{n}_\pi$	3.65	4.12	4.25	5.20
p_K	$\sim 14\%$	$\sim 10\%$	$\sim 9\%$	$\sim 6\%$

3. – Results based on Takeda's, Dyson's and Miyazawa's theories.

According to DYSON and TAKEDA, the second maximum of the total cross-section of the pion-nucleon scattering is explained by the assumption that there is a resonant state of two pions which collide at the relative kinetic energy of ~ 150 MeV. Thus the total energy (including the rest energies) of this state is $\sim 3m_\pi c^2$. The isobar state of two pions predicated by MIYAZAWA has also the energy equal to $\sim 3m_\pi c^2$. The isotopic spin and orbital angular momentum of the isobar state are chosen to be $(I=1, l=1)$ by TAKEDA and $(I=0, l=0)$ or $(I=0, l=2)$ by DYSON. In the strong coupling theory $(I=2. l=2n)$ are allowed in general.

The energies of isobaric states in the cases of TAKEDA, DYSON and MIYAZAWA are lower than that used in the last section. Thus it may be expected

that the multiplicity of pions becomes larger than that in the case of the last section.

We shall calculate, therefore, the pion multiplicity on the basis of Takeda's Dyson's and Miyazawa's models. For simplicity, we postulate the total energy of the unstable system composed of two pions to be equal to the mass of a K-meson which is equal to $3.5 m_\pi$. We also use the same approximation as in the preceeding section to evaluate the phase space integral. The isotopic spin and the angular momentum of the resonant state are taken to be ($I=1$, $l=1$), ($I=0$, $l=0$), ($I=0$, $l=2$) and ($I=2$, $l=0$). The results are shown in Table IV.

The pion multiplicities are slightly larger than that given in the last section, It is, however, still less than the observed one. If we take ($I=2$, $l=2$) and ($I=1$, $l=3$), the mean numbers of the produced pions are 4.4 and 4.5 respectively and become close to that given by EKSPONG [4].

TABLE IV.

(I, l)	$\bar{n}_\pi$	P_K
(0, 0)	3.25	$\sim 21\%$
(0, 2)	3.95	$\sim 10\%$
(1, 1)	4.01	$\sim 5\%$
(2, 0)	3.50	$\sim 17\%$
(2, 2)	4.40	$\sim 3\%$
(1, 3)	4.50	$\sim 3\%$

4. – Discussion.

The effect of the strong pion-pion interaction may be considered to appear in the angular correlation and the momentum distribution of pions in the final states. However, if we consider all pions produced in various reactions, i.e. $N+\bar{N} \to 2\pi$, 3π,..., etc., the effect of the pion interaction is smeared out and we may not point out its effect in the angular correlation and the momentum distribution. Hence, even if the angular correlation and the momentum distribution of all pions produced in various reactions are consistent with the statistical theory without any final interaction, it cannot be concluded that there is no strong pion-pion interaction.

In the special processes, for example, such as

$$\bar{p} + p \to \pi^+ + \pi^- + \pi^0 \tag{6}$$

the momentum distribution of a special pion (positive or negative charged, or neutral) has to strongly deviate from that given by Fermi's theory if there

is a strong pion-pion interaction. In the case that the isotopic spin of a system composed of two pions is zero, only the combination of (π^+, π^-) is possible. If it is 1 or 2, combinations of (π^+, π^0), (π^+, π^-) and (π^-, π^0) are possible. Thus if only the distribution of $Q_{\pi^+\pi^-}$ (the relative kinetic energy between a positive and a negative pion) in the reaction (6) deviates from that of Fermi's theory and has a peak at a particular energy, then we can conclude that there is a resonant state of two pions having the isotopic spin zero. On the other hand, if $Q_{\pi^\pm,\pi^0}$ and Q_{π^+,π^-} have a peak at the same energy, the presence of an isobar state having the isotopic spin 1 or 2 is confirmed, If, in the reaction $\bar{\mathrm{p}} + \mathrm{d} \to \pi^- + \pi^+ + \pi^-$ or $\bar{\mathrm{n}} + \mathrm{p} \to \pi^+ + \pi^+ + \pi^-$, $Q_{\pi^-\pi^-}$ or $Q_{\pi^+\pi^+}$ has a peak in a special energy, the isotopic spin of the resonant state can be considered to be 2. The rough features of the reaction $\bar{\mathrm{p}} + \mathrm{n} \to \pi^+ + \pi^- + \pi^-$ may be obtained from the reaction $\bar{\mathrm{p}} + \mathrm{d} \to \mathrm{p} + \pi^+ + \pi^- + \pi^-$, though complex correlations between the final proton and pions may occur.

In our rough estimation, it seems that though the effect of the strong pion-pion interaction is important in the process of nucleon-antinucleon annihilation, it is very hard to obtain a pion multiplicity large enough to explain the experiment if only one resonant state is effective. To get the observed multiplicity by means of only one resonant state, we must take the spin of the state to be 2 or more in the case of $Q_{\pi\pi} \sim 150$ MeV unless the interaction volume is taken to be larger than $\sim (4\pi/3)(\hbar/m_\pi c)^3$. In the above investigation, we neglected the contributions of non-resonant interactions and did not consider the case in which there were many resonant states. It is, however, not absolutely excluded that such contributions are unimportant.

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RIASSUNTO (*)

Si studia la produzione multipla dei pioni nel processo di annichilazione antinucleone-nucleone per mezzo della teoria statistica dell'interazione forte pione-pione. Le caratteristiche dello stato isobarico di due pioni si determinano in base alla distribuzione dell'impulso di un nucleone osservata nella reazione $\pi^- + \mathrm{p} \to \mathrm{N} + 2\pi$ a 1 GeV o dalle teorie di Takeda, Dyson e Miyazawa. Il numero medio dei pioni prodotti è $3 \div 4$, alquanto inferiore a quanto risulta dall'esperienza.

(*) *Traduzione a cura della Redazione.*

A Study of Einstein's Equations of Unified Field.

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Summary. — Only a few months before his death, Einstein revised his equation of unified field theory of relativity without assuming any *a priori* given condition. He considered these equations as more transparent than the previous ones. These equations will be studied and solved in tensorial form in a series of papers. In this paper, which is the first of the series, the form of the solution of the first system of equations is obtained and it has been shown that the linearized forms of these equations is the same as those of previous ones. Another set of field equations has also been obtained and studied on the same lines.

1. – Introduction.

Only a few months before his death, Einstein [1] revised his field equations of the unified field of gravitation and electromagnetism, taking the non-symmetric affine connections Γ^k_{ij} as the fundamental variables of the field and introducing the non-symmetric tensor density $\mathfrak{g}^{ij}$ of weight unity. The contracted curvature tensor in this case, is given by

$$R_{ij} = \Gamma^s_{ij,s} - \Gamma^s_{is,j} - \Gamma^s_{it}\Gamma^t_{sj} + \Gamma^s_{ij}\Gamma^t_{st} \,. \tag{1.1}$$

This tensor is not transposition symmetric. But a transposition symmetric tensor could be obtained by introducing a new pseudo tensor defined as

$$U^l_{ik} = \Gamma^l_{ik} - \Gamma^t_{it}\delta^l_k \,. \tag{1.2}$$

The contracted curvature tensor has, then, the expression

$$S_{ik}(= R_{ik}) \equiv U^s_{ik,s} - U^s_{it}U^t_{sk} + \tfrac{1}{3}U^s_{is}U^t_{tk} \,, \tag{1.3}$$

which is transposition symmetric.

[1] A. Einstein: *Meaning of Relativity*, fifth edition (Princeton, 1955), App. II.

Applying a variational principle to the scalar density

$$\mathfrak{H} = \mathfrak{g}^{ik} S_{ik}$$

he obtained the field equations in the form

$$(1.4) \quad \begin{cases} (a) \quad N^{ik}_{\ .l} \equiv \mathfrak{g}^{ik}_{\ .l} + \mathfrak{g}^{sk}(U^{i}_{sl} - \tfrac{1}{3} U^{t}_{st}\delta^{i}_{l}) + \mathfrak{g}^{is}(U^{k}_{ls} - \tfrac{1}{3} U^{t}_{is}\delta^{k}_{s}) = 0 , \\ (b) \quad S_{ik} = 0 . \end{cases}$$

Einstein's original field equations have been studied in great detail and solved by Prof. HLAVATÝ ([2]) and Mme. TONNELAT ([3]).

The object of this paper is to obtain the form of a solution of (1.4) *a*) if it exists, and then to obtain another set of field equations by taking the transposed of R_{ik}. A form of a solution of the new set has also been obtained. It has been shown that the linearized forms of both these systems of equations are the same as those of Einstein's ([4]) original set (EINSTEIN and STRAUS ([5])).

2. – Solutions of $N^{ik}_{\ .l} = 0$ for U^{n}_{lm}.

We shall suppose that a unique solution of $N^{ik}_{\ l} = 0$ exists giving U^{i}_{kl} or Γ^{i}_{kl}:

The tensor g^{ik} is defined by

$$\mathfrak{g}^{ik} = \sqrt{-g}\, g^{ik} ,$$

where

$$g = \det \mathfrak{g}^{ik} .$$

The covariant tensor g^{ik} is obtained by the property

$$g^{ik} g_{lk} = \delta^{i}_{l} = g^{ki} g_{kl} .$$

We, then, have

$$g = \det \mathfrak{g}^{ik} = \det g_{ik}$$

and

$$(2.1) \qquad g_{ik}\mathfrak{g}^{ik}_{\ .s} = -\frac{g_{.s}}{\sqrt{-g}} .$$

where comma (,) followed by an index indicates partial derivation.

([2]) V. HLAVATÝ: *Geometry of Einstein's Unified Field Theory* (1958).

([3]) M. A. TONNELAT: *La Théorie du Champ unifié d'Einstein et quelques-uns de ces developpements* (Paris, 1955).

([4]) A. EINSTEIN: *Meaning of Relativity*, fourth edition (Princeton, 1953), App. II.

([5]) A. EINSTEIN and E. G. STRAUS: *Annals of Mathematics* (Princeton, 1947), p. 731.

Multiplying the equation (1.4) a) by g_{ik}, and summing with respect to i and k, we have, by virtue of (2.1)

$$-\frac{4}{3}\,U^{t}_{(tl)}=\frac{g_{,l}}{g}=\partial_l \ln g\,, \tag{2.2}$$

where

$$U^{i}_{(jk)}=\tfrac{1}{2}(U^{i}_{jk}+U^{i}_{kj}) \qquad \text{and} \qquad \partial_l \equiv \frac{\partial}{\partial x^l}\,.$$

With the help of (2.2), the equation (1.4) a) assumes the form

$$g^{ik}_{\ ,s}-\tfrac{2}{3}g^{ik}U^{t}_{(ts)}+g^{pk}(U^{i}_{ps}-\tfrac{1}{3}U^{t}_{pt}\delta^{i}_{s})+g^{ip}(U^{k}_{sp}-\tfrac{1}{3}U^{t}_{tp}\delta^{k}_{s})=0\,.$$

Multiplying this equation by $g_{im}g_{lk}$ and summing with respect to i and k, we have

$$g_{lm\,s}=-\tfrac{2}{3}g_{lm}U^{i}_{(is)}+g_{im}U^{i}_{ls}+g_{li}U_{sm}-\tfrac{1}{3}g_{sm}U^{i}_{li}-\tfrac{1}{3}g_{ls}U^{i}_{im}\,. \tag{2.3}$$

If h_{lm} and k_{lm} are the symmetric and skew-symmetric parts of g_{lm} respectively, then from (2.3), we have

$$h_{lm\,s}=g_{i(m}U^{i}_{l)s}+U^{i}_{s(m}g_{l)i}-\tfrac{1}{3}g_{s(m}U^{i}_{l)i}-\tfrac{1}{3}U^{i}_{i(l}g_{m)s}-\tfrac{2}{3}h_{lm}U^{i}_{(is)}\,. \tag{2.4}$$

Subtracting this equation from the sum of the other two equations which are obtained from it by cyclic permutation of the indices l, m, s and dividing by 2, we find

$$[s,lm]=h_{is}U^{i}_{(lm)}-2k_{i(l}T^{i}_{m)s}-\tfrac{1}{3}[h_{ms}U^{i}_{(li)}+k_{ms}T^{i}_{li}+h_{sl}U^{i}_{(im)}+k_{sl}T^{i}_{im}]\,, \tag{2.5}$$

where

$$2T_{jk}^{\ \ i}=U^{i}_{jk}-U^{i}_{ij}\,.$$

Multiplying (2.5) by h^{sn} and summing with respect to s, we have, by virtue of (2.2)

$$\left\{\begin{matrix} n \\ lm \end{matrix}\right\}=U^{n}_{(lm)}-2h^{sn}k_{i(l}T^{i}_{m)s}+\delta^{n}_{(m}\partial_{l)}\ln\sqrt{g}-\tfrac{2}{3}h^{sn}k_{s(l}T_{m)}\,,$$

where

$$T_m=T_{im}^{\ \ i}\,.$$

Hence the unique solution of (1.4) a), if it exists, is given in the form

$$U^{n}_{lm}=\left\{\begin{matrix} n \\ lm \end{matrix}\right\}+2h^{sn}k_{i(l}T_{m)s}^{\ \ \ i}-\delta^{n}_{(m}\partial_{l)}\ln\sqrt{g}+\tfrac{2}{3}h^{sn}k_{s(l}T_{m)}+T^{n}_{lm}\,.$$

or

$$U^n_{lm} = \begin{Bmatrix} n \\ lm \end{Bmatrix} + P^n_{lm} + T_{lm}{}^n \,, \tag{2.6}$$

where P^n_{lm} is symmetric in the subscripts and is given by

$$P^n_{lm} = P^n_{(lm)} = 2h^{sn}(k_{i(l}T^i_{m)s} + \tfrac{1}{3}k_{s(l}T_{m)}) - \delta^n_{(m}\partial_{l)} \ln \sqrt{g} \,. \tag{2.7}$$

From (2.7) it is easy to verify that

$$P_{slm} = h_{sn}P^n_{lm} \,,$$

is given by

$$P_{(slm)} + h_{(sm}\partial_{l)} \ln \sqrt{g} = 0 \,, \tag{2.8}$$

where

$$P_{(lsm)} = \tfrac{1}{6}(P_{slm} + P_{lms} + P_{msl} + P_{lsm} + P_{mls} + P_{sml}) = \tfrac{1}{3}(P_{slm} + P_{lms} + P_{msl}) \,. \tag{2.9}$$

3. – Solution of $N^{ik}_l = 0$ for Γ^n_{lm}.

From (1.2), it is easy to verify that

$$U^i_{ls} - \tfrac{1}{3}U^t_{lt}\delta^i_s = \Gamma^i_{ls} \tag{3.1}$$

and

$$U^k_{sm} - \tfrac{1}{3}U^t_{tm}\delta^k_s = \Gamma^k_{sm} - \Gamma^t_{st}\delta^k_m - \tfrac{2}{3}\delta^k_s S_m \,, \tag{3.2}$$

where

$$2S_m = \Gamma^t_{tm} - \Gamma^t_{mt} \,.$$

Now the equation (2.3) can be written as

$$g_{lm,s} = -\tfrac{2}{3}g_{lm}U^i_{(is)} + g_{im}(U^i_{ls} - \tfrac{1}{3}U^t_{lt}\delta^i_s) + g_{lk}(U^k_{sm} - \tfrac{1}{3}U^t_{tm}\delta^k_s) \,,$$

which by virtue of (2.2), (3.1) and (3.2) assumes the form

$$g_{lm,s} = g_{lm}\partial_s \ln \sqrt{g} + g_{im}\Gamma^i_{ls} + g_{lk}\Gamma^k_{sm} - g_{lm}\Gamma^t_{st} - \tfrac{2}{3}g_{ls}S_m \,. \tag{3.3}$$

From (3.1) and (3.2), we have

$$2U^t_{(st)} = \Gamma^t_{ts} - 4\Gamma^t_{st} = 2S_s - 3\Gamma^t_{st} \,,$$

which, in consequence of (2.2) takes the form

$$\partial_s \ln \sqrt{g} = \Gamma^t_{st} - \tfrac{2}{3} S_s .$$

With the help of this equation, the equation (3.3) becomes

(3.4) *a*) $$g_{lm.s} = g_{im}\Gamma^i_{ls} + g_{li}\Gamma^i_{sm} - \tfrac{4}{3} g_{l(m} S_{s)} ,$$

or

(3.4) *b*) $$\underset{+\,-}{g_{l\,m;s}} = -\tfrac{4}{3} g_{l(m} S_{s)} ,$$

where semi-colon (;) followed by indices denotes covariant derivation.

From (3.4), we have

$$h_{lm.s} = g_{i(m}\Gamma^i_{l)s} + \Gamma^i_{s(l} g_{m)i} - \tfrac{2}{3} h_{lm} S_s - \tfrac{2}{3} S_{(m} g_{l)s} .$$

Subtracting this equation from the sum of the other two, obtained by the cyclic permutation of l, m and s, and dividing by 2, we have

$$[s, lm] = \Gamma^i_{s(l} k_{m)i} - k_{i(l}\Gamma^i_{m)s} + h_{si}\Gamma^i_{(ml)} - \tfrac{2}{3} g_{s(l} S_{m)} ,$$

or

$$[s, lm] = -2k_{i(l} S^i_{m)s} + h_{si}\Gamma^i_{(ml)} - \tfrac{2}{3} g_{s(l} S_{m)} .$$

Multiplying this equation by h^{ns} and summing with respect to s, we have

$$\begin{Bmatrix} n \\ lm \end{Bmatrix} = -2h^{ns} k_{i(l} S^i_{m)s} + \Gamma^n_{(lm)} - \tfrac{2}{3} h^{ns} g_{s(l} S_{m)} .$$

Therefore, the unique solution of (1.4) *a*) for Γ^n_{lm}, if it exists, is given in the form

$$\Gamma^n_{lm} = \begin{Bmatrix} n \\ lm \end{Bmatrix} + 2h^{ns}(k_{i(l} S_{m)s}{}^{i} + \tfrac{1}{3} g_{s(l} S_{m)}) + S_{lm}{}^{n} ,$$

or

(3.5) $$\Gamma^n_{lm} = \begin{Bmatrix} n \\ lm \end{Bmatrix} + Q^n{}_{lm} + S_{lm}{}^{n} ,$$

where

(3.6) $$Q^n{}_{lm} = 2h^{sn}[k_{i(l} S_{m)s}{}^{i} + \tfrac{1}{3} g_{s(l} S_{m)}] .$$

Putting

$$Q_{slm} = h_{sn} Q^n{}_{lm}$$

it is easy to verify that

(3.7) $$Q_{(slm)} = \tfrac{2}{3} h_{(sl} S_{m)} .$$

Thus we find that (2.6) *and* (3.5) *are the forms of the solutions of* (1.4) *a*).

4. – Equations for weak field.

To study the field equations for weak fields, we put in (2.3)

$$g_{lm} = \delta_{lm} + y_{lm} ,$$

where δ_{lm} have Galilean values i.e. $\delta_{lm} = 0$ when $l \neq m$, and $\delta_{lm} = 1$ when $l = m$ and y_{lm} are small quantities of the first order, such that squares and the products with U^{l}_{mn} are neglected. We then have

$$y_{lm,s} = U^{m}_{ls} + U^{l}_{sm} - \tfrac{1}{3}\delta_{sm}U^{t}_{lt} - \tfrac{1}{3}\delta_{ls}U^{t}_{tm} - \tfrac{2}{3}\delta_{lm}U^{t}_{(ts)} , \tag{4.1}$$

whence, we have

$$y_{[lm],s} = T_{sm}{}^{l} + T_{ls}{}^{m} - \tfrac{2}{3}\delta_{s[m}T_{l]}{}^{t}{}_{t} .$$

Putting $m = s$ in this equation and summing with respect to s, we have

$$\boxed{y_{[ls],s} = 0} \tag{4.2}$$

Putting $m = l$ and summing with respect to 1, we have

$$y_{ll,s} + \tfrac{4}{3}U^{l}_{(ls)} = 0 . \tag{4.3}$$

From (4.1), we have, by virtue of (4.3),

$$U^{m}_{ls} = \tfrac{1}{2}(y_{lm,s} + y_{ms,l} - y_{sl,m}) - \tfrac{1}{4}(\delta_{ms}y_{tt,l} + \delta_{lm}y_{tt,s}) + \tfrac{2}{3}\delta_{m[s}T_{l]} , \tag{4.4}$$

where

$$T_{l} = T_{t}{}^{t}{}_{l} .$$

We procced now to the field equations (1.4) b), which to the first approximation can be written as

$$U^{s}_{ik,s} = 0 ,$$

or substituting from (4.4), we have

$$\tfrac{1}{2}(y_{lm,sn} + y_{ms,ln} - y_{sl,mn}) - \tfrac{1}{2}y_{tt,ls} + \tfrac{2}{3}T_{[l,s]} = 0 .$$

The symmetric part of this equation yields

$$y_{(lm),sn} + y_{(sn),lm} - y_{(sl),mn} - y_{mn,ls} = 0 . \tag{4.5}$$

The skew-symmetric part of this equation yields

$$\tfrac{1}{2}[y_{[ms],lm} + y_{[lm],sm} - y_{[sl],mm}] + \tfrac{2}{3}T_{[l,s]} = 0 .$$

Differentiating this equation with respect to x^n, we have

$$y_{[ms],nlm} + y_{[lm],nsm} - y_{[sl],nmm} + \tfrac{4}{3}T_{[l,s]n} = 0 .$$

Adding to this equation the sum of the other two equations obtained by the cyclic permutation of l, n, and s, we have

$$\boxed{y_{l[sl],nmn} + y_{[ln],smn} + y_{[ns],lmn} = 0} \tag{4.6}$$

The linearized forms (4.2), (4.5) and (4.6) of the new set of Einstein's field equations are the same as those of the original set (EINSTEIN and STRAUS [5]).

5. – Another set of field equations.

To obtain another set of field equations of the unified field by the variational principle, we start with the tensor $\widetilde{R}_{ik}$ transposed to R_{ik}, i.e.

$$\widetilde{R}_{ik} = \Gamma^s_{ki,s} - \Gamma^s_{si,k} - \Gamma^s_{ti}\Gamma^t_{ks} + \Gamma^s_{ki}\Gamma^t_{ts} . \tag{5.1}$$

This tensor also is not transposition symmetric. But a transposition symmetric tensor can be obtained by introducing another new pseudo tensor V^l_{ik} defined by

$$V^l_{ki} = \Gamma^l_{ki} - \Gamma^t_{ti}\delta^l_k . \tag{5.2}$$

The tensor (5.1) then assumes the form

$$\widetilde{S}_{ik}(\equiv \widetilde{R}_{ik}) = V^s_{ki,s} - V^s_{ti}V^t_{ks} + \tfrac{1}{3}V^s_{si}V^t_{kt} , \tag{5.3}$$

which is obviously, transposition symmetric.

We introduce the scalar density

$$\overline{\mathfrak{h}} = \mathfrak{g}^{ik}\widetilde{S}_{ik}$$

and apply the variational principle

$$\delta\int\overline{\mathfrak{h}} - \mathrm{d}\tau = 0 . \tag{5.4}$$

It is easy to verify that the integral is transformation invariant. Also the integral is invariant with respect to $\bar{\lambda}$-transformation

$$\overset{*}{\Gamma}{}^{i}_{jk} = \Gamma^{i}_{jk} + 2\delta^{i}_{k}\bar{\lambda}_{,j} .$$

It may be noted that the integral of Einstein's theory corresponding to the variational principle

$$\delta\int \mathfrak{h}\, d\tau = 0 ,$$

is invariant with respect to λ-transformation

$$\overset{*}{\Gamma}{}^{i}_{jk} = \Gamma^{i}_{jk} + 2\delta^{i}_{j}\lambda_{,k} .$$

By variation of $\bar{\mathfrak{h}}$ with respect to $\mathfrak{g}$ and V, we find

$$\delta\bar{\mathfrak{h}} = \bar{S}_{ik}\,\delta\mathfrak{g}^{ik} - \delta V^{l}_{ki}[\mathfrak{g}^{ik}_{,l} + \mathfrak{g}^{sk}(V^{i}_{ls} - \tfrac{1}{3}V^{t}_{ts}\delta^{i}_{l}) + \mathfrak{g}^{is}(V^{k}_{sl} - \tfrac{1}{3}V^{t}_{ts}\delta^{k}_{l}) + (\mathfrak{g}^{ik}\,\delta V^{s}_{ki})_{,s} .$$

The variations of $\mathfrak{g}^{ik}$ and V^{l}_{ki}, which vary independently, vanish at the boundary of the domain of integration. The equation (5.4) can be written as

$$\int \delta\bar{\mathfrak{h}}\, d\tau = 0 .$$

Now, as δV^{l}_{ki} vanishes at the boundary, the term $(\mathfrak{g}^{ik}\delta V^{s}_{ki})_{,s}$ does not give any contribution. Therefore, we have the field equations

$$\text{(5.5)} \qquad \begin{cases} a)\ \ \overline{N}^{ik}_{\ \ l} = \mathfrak{g}^{ik}_{\ \ ,l} + \mathfrak{g}^{sk}(V^{i}_{ls} - \tfrac{1}{3}V^{t}_{ts}\delta^{i}_{l}) + \mathfrak{g}^{is}(V^{k}_{sl} - \tfrac{1}{3}V^{t}_{st}\delta^{k}_{l}) = 0 \\ b)\ \ \bar{S}_{ik} = 0 . \end{cases}$$

Multiplying the equation (5.5) a) by g_{ik} and summing with respect to i, and k, we have

$$\text{(5.6)} \qquad \partial_{l} \ln \sqrt{g} + \tfrac{2}{3}V^{t}_{(tl)} = 0 .$$

Proceeding as in Sect. **2**, we find that a solution of (5.5) a) is in the form

$$\text{(5.7)} \qquad V^{n}_{.\,l} = \left\{ {n \atop ml} \right\} + R^{n}_{\ ml} + N_{ml}{}^{n} ,$$

where

$$N^{n}_{n\,l} = V^{n}_{[ml]} ,$$

and

$$N_{m} = N_{nm}{}^{n} ,$$

$$\text{(5.8)} \qquad R^{n}_{\ lm} = R^{n}_{\ (lm)} = -2h^{sn}(k_{i(l}N^{i}_{m)s} + \tfrac{1}{3}k_{s(l}N_{m)}) - \delta^{i}_{m}\partial_{l} \ln \sqrt{g} .$$

It is again easy to verify that

$$R_{(slm)} + h_{(sm}\partial_{l)} \ln \sqrt{g} = 0 .$$

Putting the value of $V^i{}_{ls}$ in (5.5) a) and multiplying this equation by $g_{im}g_{sk}$, it takes the form

(5.9) a) $$g_{lm,s} = g_{im}\Gamma^i_{sl} + g_{li}\Gamma^i_{ms} + \tfrac{4}{3}g_{l(m}S_{s)} ,$$

or

(5.9) b) $$\underset{-+}{g_{l\,m\,s}} = \tfrac{4}{3}g_{l(m}S_{s)} .$$

A unique solution of this equation, if it exists, is given in the form

(5.10) $$\Gamma_l{}^n{}_m = \left\{ \begin{matrix} n \\ lm \end{matrix} \right\} - Q^n{}_{lm} + S_{lm}{}^n .$$

6. – Bianchi identities in the new field.

It has been remarked before that $\int \tilde{\mathfrak{H}}\, d\tau$ is invariant with respect to infinitesimal co-ordinate transformations and to $\bar{\lambda}$-transformations. We shall use these to derive the identities between the field equations.

We shall first take the infinitesimal co-ordinate transformation

(6.1) $$\bar{x}^k = x^k + \xi^k ,$$

where ξ^k is an arbitrary infinitesimal vector.

From the invariance of $\int \tilde{\mathfrak{H}}\, d\tau$, it follows that its variation vanishes identically if one inserts in $d\tilde{\mathfrak{H}}$ the variations $\delta \mathfrak{g}^{ik}$ and $\delta V^{ik}{}_l$.

Now, for the infinitesimal co-ordinate transformation

(6.2) $$\delta \mathfrak{g}^{ik} \equiv - L\mathfrak{g}^{ik} = \mathfrak{g}^{lk}\xi^i{}_{,l} + \mathfrak{g}^{im}\xi^k{}_{,m} - \mathfrak{g}^{ik}\xi^s{}_{,s} ,$$

where L is the operator for the Lie-derivative (Yano and Bochner (6)).

Using the law of transformation for the connections it can be easily verified that the law of transformation for $V^i{}_{jk}$ is

$$\bar{V}^l{}_{ik} = V^a_{bc}\frac{\partial \bar{x}^l}{\partial x^a}\frac{\partial x^b}{\partial \bar{x}^i}\frac{\partial x^c}{\partial \bar{x}^k} - \frac{\partial^2 \bar{x}^l}{\partial x^s \partial x^t}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^t}{\partial \bar{x}^k} + \bar{\delta}^l_i \frac{\partial^2 \bar{x}^t}{\partial x^s \partial x^p}\frac{\partial x^s}{\partial \bar{x}^t}\frac{\partial x^p}{\partial \bar{x}^k} ,$$

(6) K. Yano and S. Bochner: *Curvature and Betti Numbers* (Princeton, 1953).

whence, by straightforward calculations, it can be verified that

$$\delta V^l_{ik} \equiv -LV^l_{ik} = V^s_{ik}\xi^l_{,s} - V^l_{sk}\xi^s_{,i} - V^l_{is}\xi^s_{,k} - \xi^l_{,ik} - V^l_{ik,s}\xi^s + \delta^l_i\xi^t_{,tk} . \tag{6.3}$$

Inserting these variations in $\delta\bar{\mathfrak{h}}$, and choosing ξ^i such that it, together with its first derivative, vanishes at the boundary, we have that the integral

$$\int (\widetilde{S}_{ik}\delta\mathfrak{g}^{ik} - \overline{N}^{ik}_{\ \ l}\delta V^l_{ik})\,\mathrm{d}\tau \tag{6.4}$$

vanishes identically.

Substituting the values of $\delta\mathfrak{g}^{ik}$ and δV^l_{ik} from (6.2) and (6.3) in (6.4), we have

$$\int [(\widetilde{S}_{\lambda k}\mathfrak{g}^{sk} + \widetilde{S}_{i\lambda}\mathfrak{g}^{is} - \delta^s_\lambda\widetilde{S}_{ik}\mathfrak{g}^{ik} - \overline{N}^{ki}_{\ \ \lambda}V^s_{ik} + \overline{N}^{ks}_{\ \ l}V^l_{\lambda k} + \overline{N}^{si}_{\ \ l}V^l_{i\lambda})\xi^\lambda_{,s} -$$
$$- (\widetilde{S}_{ik}\mathfrak{g}^{ik}_{\ \ ,\lambda} + \overline{N}^{ki}_{\ \ l}V^l_{ik,\lambda})\xi^\lambda - \widetilde{N}^{ki}_{\ \ \lambda}\xi^\lambda_{,ik} + \overline{N}^{kl}_{\ \ l}\xi^\lambda_{,\lambda k}]\,\mathrm{d}\tau = 0 .$$

Integrating partially the first term once, and the third and fourth terms twice. we have, in consequence of the restrictions on ξ^λ

$$\int [(\widetilde{S}_{\lambda k}\mathfrak{g}^{sk} + \widetilde{S}_{i\lambda}\mathfrak{g}^{is} - \delta^s_\lambda\widetilde{S}_{ik}\mathfrak{g}^{ik} - \overline{N}^{ki}_{\ \ \lambda}V^s_{ik} + \overline{N}^{ks}_{\ \ l}V^l_{\lambda k} + \overline{N}^{si}_{\ \ l}V^l_{i\lambda})_{,s} +$$
$$+ (\widetilde{S}_{ik}\mathfrak{g}^{ik}_{\ \ ,\lambda} + \overline{N}^{ki}_{\ \ l}V^l_{ik,\lambda}) - \overline{N}^{ki}_{\ \ \lambda,ik}\,\overline{N}^{kl}_{\ \ l,\lambda k}]\xi^\lambda\,\mathrm{d}\tau = 0 ,$$

which can be written as

$$\int B_\lambda\xi^\lambda\,\mathrm{d}\tau = 0 ,$$

where B_λ, the expression in square brackets, is of the first order in $\widetilde{S}_{ik}$ and second order in $N^{ik}_{\ \ l}$. Since ξ^λ is taken arbitrarily, we have

$$\boxed{B_\lambda = 0} \tag{6.5}$$

These are four identities that hold between the left hand members of the field equations (5.5) and correspond to Bianchi identities in this field.

Now we will establish the fifth identity corresponding to the invariance of $\int\bar{\mathfrak{h}}\,\mathrm{d}\tau$ with respect to infinitesimal $\bar{\lambda}$-transformations. In this case

$$\delta\mathfrak{g}^{ik} = 0$$

and

$$\delta V^l_{ki} = 2(\delta^l_i\lambda_{,k} - \delta^l_k\lambda_{,i}) ,$$

where λ is infinitesimal and vanishes at the boundary of the domain of integration.

We, therefore, have

$$\int \overline{N}^{ik}{}_{l}(\delta^{l}_{i}\lambda_{,k} - \delta^{l}_{k}\lambda_{,i})\,\mathrm{d}\tau = 0\,.$$

Integrating this equation by parts, we have

$$2\int \overline{N}^{[ik]}{}_{,ki}\,\lambda\,\mathrm{d}\tau = 0\,.$$

Therefore, the desired identity is

$$\boxed{\overline{N}^{[ik]}{}_{,ki} \equiv 0}\,. \tag{6.6}$$

From (5.5) *a*) it is easy to determine that

$$\overline{N}^{[ik]}{}_{k} = \mathfrak{g}^{[ik]}{}_{,k}\,.$$

In a subsequent work, the equations (1.4) *a*) and (5.5) *a*) will be completely solved.

RIASSUNTO (*)

Appena pochi mesi prima della sua morte EINSTEIN revisionò la sua equazione della teoria relativistica dei campi unificati senza porre alcuna condizione data *a priori*. Considerò queste equazioni come più trasparenti delle precedenti. In una serie di articoli studieremo queste equazioni e le risolveremo in forma tensoriale. Nel presente lavoro, il primo della serie, troviamo la forma della soluzione del primo sistema di equazioni e dimostriamo che la forma linearizzata di tali equazioni è la stessa delle precedenti. Si ottiene, inoltre, e si studia secondo gli stessi concetti un altro sistema di equazioni.

(*) *Traduzione a cura della Redazione.*

Absorption of Slow Negative Kaons in Hydrogen and Deuterium (*).

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Summary. — The transition probabilities for the competing reactions resulting from the absorption of slow negative kaons in hydrogen and deuterium have been calculated on the basis of a simple model. Computations were carried out for scalar and pseudoscalar coupling of the kaon from the 1*s* and 2*p* orbitals of the appropriate K-mesic atom. The absorption from the 2*p* state is enhanced with respect to the pion case because of the smaller size of the orbit. This effect tends to reduce the possibility of distinguishing between the scalar and pseudoscalar character of the kaon coupling in strong interactions. The emission of a single pion is favoured for deuterium as well as for hydrogen. Qualitative conclusions are drawn concerning the relative importance of the other modes of kaon absorption in both hydrogen and deuterium.

1. – Introduction.

The study of the absorption of slow negative pions in hydrogen and deuterium was decisive in determining some of the fundamental properties of the pion, particularly its parity (1). It will be recalled that the slow negative pion reactions in hydrogen and deuterium are:

$$(1a) \qquad \pi^- + \mathrm{p} \to \mathrm{n} + \gamma \,,$$

$$(1b) \qquad \to \mathrm{n} + \pi^0 \,,$$

$$(2a) \qquad \pi^- + \mathrm{d} \to 2\mathrm{n} \,,$$

$$(2b) \qquad \to 2\mathrm{n} + \gamma \,,$$

$$(2c) \qquad \to 2\mathrm{n} + \pi^0 \,.$$

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(1) Cfr. review article by R. E. Marshak: *Rev. Mod. Phys.*, **23**, 137 (1951); also, S. Tamor: *Phys. Rev.*, **82**, 38 (1951).

It turns out that the most primitive theoretical calculations (lowest order perturbation calculations for hydrogen and the same in the impulse approximation for deuterium) for the transition probabilities of the above reactions lead to rather good agreement with the experimental results.

If one inquires into the reasons for this excellent agreement, one could perhaps enumerate them as follows:

1) Reaction (1*a*) in hydrogen is the inverse of photo-pion production which is treated rather successfully near threshold by the renormalized Born approximation (Kroll-Ruderman theorem [2]).

2) The charge exchange reaction (1*b*) in hydrogen does not involve the « meson pair » term in the Foldy-Dyson (F-D) transformed Hamiltonian which is strongly damped by higher order terms [3]; it depends on the *s*-wave isotopic spin-dependent term in the F-D Hamiltonian. This term contains the square of the same renormalized coupling constant which appears before the (*p*-wave) term which is needed [3,4] for reaction (1*a*).

3) Reaction (2*a*) in deterium effectively involves the *p*-wave pion-nucleon interaction which, on the basis of the work of CHEW and LOW, appears to give reasonable results in renormalized Born approximation as long as the energies are not too high.

4) Reactions (2*b*) and (2*c*) in deuterium depend on the validity of the impulse approximation which should be a reasonable approximation for a weakly bound nucleus like the deuteron. Except for the extra neutron in the deuteron, these two reactions are the analogs of reactions (1*a*) and (1*b*) in hydrogen and do not involve any new features.

While there is no *a priori* reason why a weak coupling calculation of the absorption of slow negative kaons in hydrogen and deuterium should be as successful as the pion case, it seems worthwhile to carry out such calculations in order to achieve some insight into any particularly interesting qualitative effects which may show up. The pion calculations will serve as guides for the kaon calculations in ways which will become apparent as each reaction is discussed separately.

The slow negative kaon reactions in hydrogen and deuterium which we consider are:

$$(3a) \qquad K^- + p \to n + K^0$$

$$(3b) \qquad \to Y + \pi$$

$$(3c) \qquad \to Y + \gamma$$

$$(3d) \qquad \to \Lambda + 2\pi$$

(2) N. M. KROLL and M. A. RUDERMAN: *Phys. Rev.*, **93**, 233 (1954).
(3) A. KLEIN: *Phys. Rev.*, **99**, 998 (1955).
(4) S. OKUBO and R. E. MARSHAK: to be published in *Annals of Physics*.

$$(4a) \qquad K^- + d \to Y + N$$

$$(4b) \qquad \to Y + N + \pi$$

$$(4c) \qquad \to Y + N + \gamma$$

$$(4d) \qquad \to \Lambda + N + 2\pi \,,$$

where Y denotes a Λ or Σ hyperon, N a nucleon (as distinct from n, the neutron) and the remaining symbols have their usual meaning. It is not expected that there will be the analog of reaction (3*a*) in deuterium (see below). If the Σ hyperon-nucleon force is strong enough to form a Σ hyper-deuteron, the bound state modes would also have to be considered for the deuterium case. Calculations of these bound state modes resulting from kaon absorption in deuterium have been presented recently by PAIS and TREIMAN ([5]). Since the existence of the Σ hyper-deuteron (the Λ hyper-deuteron probably does not exist ([6])) is conjectural, we shall restrict ourselves to the unbound modes.

There are now fairly strong arguments for spin 0 for the kaon ([7]). However, the parity of the kaon (relative to the YN pair) in strong interactions is still not fixed and indeed one purpose of the present calculation is to ascertain whether slow negative kaon absorption experiments in hydrogen and deuterium can help to decide this important property of the kaon. Moreover, the absorption processes from the $2p$ states of the kaon-hydrogen and kaon-deuterium atoms are much enhanced with respect to the corresponding processes for the pion atoms due to the smaller orbits (the kaon mass is three times the pion mass). The reactions (3*a*)–(3*d*) and (4*a*)–(4*d*) are therefore computed for scalar and pseudoscalar kaons (with respect to hyperon-nucleon pair) absorbed from both the $1s$ and $2p$ states.

2. – General formulation.

The strong interaction Hamiltonian is chosen to be of the direct Yukawa, charge independent type ([8]) (we omit all terms involving the Ξ hyperon since we calculate in lowest order perturbation theory):

$$(5) \qquad H = H_{NN\pi} + H_{\Sigma\Sigma\pi} + H_{\Sigma\Lambda\pi} + H_{N\Sigma K} + H_{N\Lambda K} \,,$$

([5]) A. PAIS and S. B. TREIMAN: *Phys. Rev.*, **107**, 1396 (1957).

([6]) R. H. DALITZ: *Proc. of Sixth Rochester Conf. on High Energy Physics* (1956).

([7]) Cfr. review article by R. H. DALITZ: *Reports on Progress in Physics*, **20**, 163 (1957) (The Physical Society, London).

([8]) A. SALAM: *Nucl. Phys.*, **2**, 173 (1956).

where

$$(5a) \qquad H_{NN\pi} = g_\pi \bar{N}(i\gamma_5)\boldsymbol{\tau}\cdot\boldsymbol{\pi}N$$

$$(5b) \qquad H_{\Sigma\Sigma\pi} = -ig_\pi \bar{\boldsymbol{\Sigma}}(i\gamma_5)\times\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$$

$$(5c) \qquad H_{\Sigma\Lambda\pi} = g_\pi[\bar{\Lambda}(i\gamma_5)\boldsymbol{\Sigma}\cdot\boldsymbol{\pi} + \bar{\boldsymbol{\Sigma}}(i\gamma_5)\Lambda\cdot\boldsymbol{\pi}]$$

$$(5d) \qquad H_{N\Sigma K} = g_K[\underline{N\boldsymbol{\tau}K}(\Gamma)\boldsymbol{\Sigma} + \bar{\boldsymbol{\Sigma}}(\Gamma)\underline{\bar{K}\boldsymbol{\tau}N}]$$

$$(5e) \qquad H_{N\Lambda K} = g_K[\underline{N\bar{K}}K(\Gamma)\Lambda + \bar{\Lambda}(\Gamma)\underline{\bar{K}N}]$$

with the letters referring to the destruction of the corresponding particles. The operator $\Gamma = 1$ for a scalar and $i\gamma_5$ for a pseudoscalar kaon. The pion and kaon coupling constants to the baryons are assumed universal. This assumption of « global symmetry » (9) is made to reduce the arbitrariness in the numerical computations; the dependence, if any, of our qualitative conclusions on this assumption is discussed at the end. The Hamiltonian (5) will suffice to calculate all of the reactions (3*a*)–(4*d*) except (3*c*) and (4*c*) for which it will be necessary to add the appropriate terms involving the photon field.

We can easily see that if the orbital of the kaon is expanded in plane waves, the amplitude is mostly concentrated near zero momentum and its width is of the order of $h/a_0 = (e^2/hc)mc$, a_0 being the kaon Bohr radius (m is the kaon mass). This is quite small compared to the characteristic momentum in any of the reactions (except possibly (3*a*)). This suggests that we treat the absorption problem from an atomic orbital as if it were an absorption process from a scattering state with vanishing kinetic energy.

The transition operator T for the absorption of a zero energy kaon by a single baryon is easily calculated in lowest order, including all the relativistic effects of intermediate states, with the aid of Feynman diagrams. T is an explicit function of the initial and final baryon 4-momenta P_i and P_f, of the kaon 4-momentum q and of the 4-momentum of the finally produced quantum k. While T is a 4×4 matrix, the transition matrix element for positive to positive energy states of a baryon is given by evaluating the 2×2 matrix

$$\sqrt{\frac{E_{p_i}+M_i}{2E_{p_i}}}\sqrt{\frac{E_{p_f}+M_f}{2E_{p_f}}}\left(1, \frac{-(\boldsymbol{\sigma}\cdot\boldsymbol{P})}{E_{p_f}+M_f}\right)T_{\left(1,\ (\boldsymbol{\sigma}\cdot\boldsymbol{P}_i)/(E_{p_i}+M_i)\right)} \equiv T\,,$$

between Pauli spinors. The 4×4 and 2×4 matrices on both sides of T reduce it to a 2×2 matrix, to match the Pauli spinors. Assuming that the 3-momenta $\boldsymbol{P}_f$, $\boldsymbol{P}_i$, $\boldsymbol{k}$, $\boldsymbol{q}$ are small compared to the baryon mass, we can expand T in powers of the 3-momenta and express it as an explicit function of $\boldsymbol{P}_i$, $\boldsymbol{P}_f$, $\boldsymbol{k}$, $\boldsymbol{q}$, $\boldsymbol{\sigma}$.

(9) M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

The transition operator θ is

$$\theta = \frac{1}{\sqrt{2\omega_k}} \cdot \frac{\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]}{\sqrt{V}} T(\boldsymbol{P}_i, \boldsymbol{P}_f, \boldsymbol{k}, \boldsymbol{q}, \boldsymbol{\sigma}) \cdot \frac{1}{\sqrt{2m}} \cdot \varphi_{nl}(\boldsymbol{r}) , \tag{6}$$

where $\boldsymbol{r}$ is the co-ordinate of the absorbing nucleon and φ_{nl} is the normalized atomic orbital; $\boldsymbol{q}$ is in fact a gradient operator on φ_{nl}, namely

$$\boldsymbol{q}\varphi_{nl} \to -i\boldsymbol{\nabla}\cdot\varphi_{nl}$$

evaluated at the position of the absorbing nucleon. It is shown below that

$$\begin{cases} \varphi_{1s}(0) = \text{const} & (\boldsymbol{\nabla}\varphi_{1s})_0 \cong 0 \\ \varphi_{2p}(0) = 0 & (\boldsymbol{\nabla}\varphi_{2p})_0 \cong \text{const} , \end{cases} \tag{7}$$

so that the terms linear in $\boldsymbol{q}$ are effective for the absorption from the $2p$ orbital and the terms independent of $\boldsymbol{q}$ are responsible for the absorption from the $1s$ orbital. We take the system of units $c = \hbar = M_0 = 1$ throughout, and let M, M_0, m, μ denote the hyperon (Λ or Σ), nucleon, kaon and pion masses respectively.

The absorption of slow negative kaons in deuterium is calculated in the impulse approximation. Hence the transition operator, which describes the annihilation of the kaon and the creation of the other quanta and the conversion of the deuteron into the final hyperon-nucleon system, is essentially the same as for the one-baryon problem, i.e., the absorption in hydrogen. Furthermore, the kaon is bound to the center of mass of the deuteron in the K-deuterium atom, but its Bohr radius is about 15 times the radius of the deuteron. Therefore, the atomic orbital φ_{nl} and its derivative $\boldsymbol{\nabla}\cdot\varphi_{nl}$ can be replaced by their values at the origin as in the case of hydrogen.

3. – Kaon absorption in hydrogen.

3·1. *Mesonic absorption in hydrogen.* – We first consider reaction (3*b*). The Q-value is 177 MeV for Λ production and 105 MeV for Σ production. The transition matrix must be pseudoscalar if the kaon is scalar and scalar if the kaon is pseudoscalar. The invariants are formed by the available vector quantities $\boldsymbol{k}$, $\boldsymbol{q}$, $\boldsymbol{\sigma}$:

$$T(\text{S}) = f_1(\boldsymbol{\sigma}\cdot\boldsymbol{k}) + f_2(\boldsymbol{\sigma}\cdot\boldsymbol{q}) ,$$

$$T(\text{PS}) = g_1 + g_2\boldsymbol{k}^2 + g_3(\boldsymbol{k}\cdot\boldsymbol{q}) + g_4(\boldsymbol{\sigma}\cdot\boldsymbol{k}\times\boldsymbol{q}) ,$$

where f's and g's are constants if terms higher than the second power of momentum are discarded. The perturbation diagrams for this process are shown in Fig. 1, where (c) shows the intermediate formation of a different kind of hyperon. Straightforward computation gives the coefficients f and g as explicit algebraic functions of the masses of the participating particles with the results given in Table I.

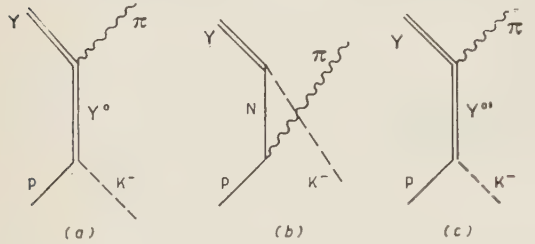

Fig. 1. – Feynman diagrams for mesonic absorption of kaons in hydrogen.

TABLE I. – f *and* g ($c = h = M_0 = 1$).

Process	f_1	f_2	g_1	g_2	g_3	g_4
$(\Lambda^0\pi^0)$	0.421	0.123	— 0.960	0.559	0.363	$-1.845i$
$(\Sigma^+\pi^-)$	0.348	— 0.048	— 0.010	0.008	0.172	$0.172i$
$(\Sigma^0\pi^0)$	1.231	0.134	— 0.944	0.601	0.734	$-1.800i$
$(\Sigma^-\pi^+)$	2.176	0.212	— 1.877	1.200	1.296	$-3.772i$

It is seen from Table I that there is a cancellation between the diagrams (a) and (c) for the process $(\Sigma^+\pi^-)$. This result does not depend on the fact that we have calculated in lowest order perturbation theory if the mass difference of the Λ and Σ is neglected and equal coupling constants for (NΛK) and (NΣK) are assumed. To show this we rewrite the interaction Hamiltonian [5] in the form:

$$H_\pi = g_\pi(\underline{\bar{\mathrm{N}}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,\mathrm{N}} + \underline{\bar{\mathrm{Y}}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,\mathrm{Y}} + \underline{\bar{\mathrm{Z}}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,\mathrm{Z}})\,, \tag{8a}$$

$$H_k = \sqrt{2}\,g_k(\underline{\bar{\mathrm{Y}}\mathrm{N}}\,\bar{\mathrm{K}}^0 + \underline{\mathrm{Z}\bar{\mathrm{N}}}\,\bar{\mathrm{K}}^+ + \underline{\bar{\mathrm{N}}\mathrm{Y}}\,\mathrm{K}^0 + \underline{\bar{\mathrm{N}}\mathrm{Z}}\,\mathrm{K}^+)\,, \tag{8b}$$

where Y and Z are Gell-Mann's isospinors [9]:

$$Y = \begin{pmatrix} \Sigma^+ \\ \frac{1}{\sqrt{2}}(\Lambda^0 - \Sigma^0) \end{pmatrix}, \qquad Z = \begin{pmatrix} \frac{1}{\sqrt{2}}(\Lambda^0 + \Sigma^0) \\ \Sigma^- \end{pmatrix}, \tag{9}$$

and K^+, K^0 are the components of the isospinor K. The absorption of the negative kaon takes place through $\bar{\mathrm{Z}}\,\underline{\mathrm{N}}\,\bar{\mathrm{K}}^+$. Z thus formed never goes to Y by any repeated interactions with pions and kaons. Hence the upper component of Y, Σ^+ will not be found in the final state and this channel should be inhibited compared to $(\Sigma^-\pi^+)$, for example.

The transition rate (transition probability per unit time) reads

$$w = \frac{g_{\bar{K}}^2}{4\pi} \cdot \frac{g_\pi^2}{4\pi} \cdot \tau \cdot k \, |\mathfrak{M}|^2 , \tag{10}$$

where the characteristic reciprocal time is

$$\tau_{1s} = \frac{M^2 + (M_0 + m)^2 - \mu^2}{2m(M + m)} \cdot \frac{4}{a_0^3} \cdot M_0^{-1} = \begin{cases} 1.38 \cdot 10^{17} \text{ s}^{-1} & (\Lambda) , \\ 1.41 \cdot 10^{17} \text{ s}^{-1} & (\Sigma) , \end{cases}$$

$$\tau_{2p} = \frac{M^2 + (M_0 + m)^2 - \mu^2}{2m(M + m)} \frac{1}{24 a_0^5} M_0^{-3} = \begin{cases} 9.11 \cdot 10^9 \text{ s}^{-1} & (\Lambda) , \\ 9.27 \cdot 10^9 \text{ s}^{-1} & (\Sigma) , \end{cases}$$

and

$$|\mathfrak{M}(\mathrm{S},\ 1s)|^2 = |f_1|^2 k^2 \tag{11a}$$

$$|\mathfrak{M}(\mathrm{PS},\ 1s)|^2 = |g_1 + g_2 k^2|^2 \tag{11b}$$

$$|\mathfrak{M}(\mathrm{S},\ 2p)|^2 = 3\,|f_2|^2 \tag{11c}$$

$$|\mathfrak{M}(\mathrm{PS},\ 2p)|^2 = (|g_3|^2 + 2\,|g_4|^2) k^2 , \tag{11d}$$

with the kinematics giving

$$k^2 = \frac{(M_0 + m + M + \mu)(M_0 + m + M - \mu)(M_0 + m - M + \mu)(M_0 + m - M - \mu)}{4(M_0 + m)^2} =$$

$$= \begin{cases} 0.0726 M_0^2 & (\Lambda) , \\ 0.0383 M_0^2 & (\Sigma) . \end{cases}$$

The transition rates (in s^{-1}) are listed in Table II.

TABLE II. – *Transition rate for mesonic absorption in* s^{-1}.

Process		(S, 1s)	(PS, 1s)	(S, 2p)	(PS, 2p)
$(\Lambda^0\pi^0)$	$(g_k^2/4\pi)(g_\pi^2/4\pi)\times$	$4.79 \cdot 10^{14}$	$3.14 \cdot 10^{16}$	$1.09 \cdot 10^8$	$1.23 \cdot 10^9$
$(\Sigma^+\pi^-)$	» ×	$1.28 \cdot 10^{14}$	$2.86 \cdot 10^{12}$	$1.25 \cdot 10^7$	$6.17 \cdot 10^6$
$(\Sigma^0\pi^0)$	» ×	$1.58 \cdot 10^{15}$	$2.33 \cdot 10^{16}$	$5.83 \cdot 10^7$	$4.77 \cdot 10^8$
$(\Sigma^-\pi^+)$	» ×	$4.99 \cdot 10^{15}$	$9.22 \cdot 10^{16}$	$2.44 \cdot 10^8$	$2.09 \cdot 10^9$

The difference by more than a factor 10 between 1*s* absorption for a pseudoscalar kaon and that for a scalar kaon is due to the fact that a *s*-wave pion is emitted in the former case whereas a *p*-wave pion is emitted in the latter.

The optical transition rate

$$w_{2p\to 1s} = 3.98\cdot 10^{11}\ \mathrm{s}^{-1}\,,$$

will now be compared to the total absorption rate from the $2p$ orbital. From the preliminary photo-kaon production experiments ([10]), the kaon coupling constant is of the order:

$$\frac{g_K^2}{4\pi} = 1 \div 3 \qquad \text{for a scalar kaon}\,,$$

$$\frac{g_K^2}{4\pi} = 3 \div 10 \qquad \text{for a pseudoscalar kaon}\,.$$

With these values and taking $g_\pi^2/4\pi = 15$, we note that for a scalar kaon the total absorption from the $2p$ orbital is more than a factor 10 smaller than the optical transition rate, while for a pseudoscalar kaon the $2p$ absorption is comparable to the optical transition rate.

It is seen from Table II that the Σ^- channel is the most favored in all cases, the Σ^0 channel is favored over the Λ^0 channel in the (S, $1s$) case and is less than (although comparable to) the Λ^0 channel in the other cases, and the Σ^+ channel is greatly suppressed under our assumption of « global symmetry ». The preliminary experimental branching ratios are ([11]):

$$W(\Lambda^0\pi^0) : W(\Sigma^+\pi^-) : W(\Sigma^0\pi^0) : W(\Sigma^-\pi^+) = 1 : 4 : 4 : 8\,.$$

One might be tempted to conclude from these numbers that the kaon is scalar and is absorbed from the $1s$ orbital and that « global symmetry » is a poor assumption; this, of course, would be premature. However, it is encouraging that the dominant channel has larger isotopic spin factors associated with it, rather independent of the parity of the kaon or of the orbital from which it is absorbed.

Another very interesting feature of the experimental results is the apparent suppression of the ($\Lambda^0\pi^0$) mode. One might argue from this that the (NΛK) coupling constant is smaller than the (NΣK) coupling constant. As a limiting case, we have assumed that the (NΛK) coupling is absent but that global symmetry still holds. The resulting branching ratios are given in Table II*a*.

([10]) P. L. Donoho and R. L. Walker: *Phys. Rev.*, **107**, 1198 (1957); A. Silverman, R. R. Wilson and W. M. Woodward: *Phys. Rev.*, **108**, 501 (1957).

([11]) L. W. Alvarez, H. Bradner, P. Falk-Vairant, J. D. Gow, A. H. Rosenfeld, F. T. Solmitz and R. D. Tripp: UCRL-3775 (July 1957).

TABLE IIa. – *Branching ratios for* $g_{N\Lambda K} = 0$.

Process	(S, 1s)	(PS, 1s)	(S, 2p)	(PS, 2p)	Exper.
$(\Lambda^0\pi^0)$	1	1	1	1	1
$(\Sigma^+\pi^-)$	0.46	1.06	1.01	0.48	4
$(\Sigma^0\pi^0)$	1.20	2.57	2.58	1.26	4
$(\Sigma^-\pi^+)$	2.28	15.8	4.90	5.81	8

The branching ratios will also change if global symmetry is violated and the pion-hyperon couplings are weaker than the pion-nucleon coupling. Again, as a limiting case, we assume that the $(Y Y \pi)$ couplings are zero; then only diagram (*b*) in Fig. 1 contributes. The branching ratios are then predicted to be $\lambda^2 : 0 : 1 : 4$, where λ is the ratio of the $(N\Lambda K)$ and $(N\Sigma K)$ coupling constants, for both parities of the kaon and for absorption from the 1*s* and 2*p* orbitals.

3·2. *Other modes in hydrogen.* – The other three slow negative kaon reactions in hydrogen are calculated in a fashion analogous to mesonic absorption; an invariant transition operator is constructed out of the available vectors and the lowest order perturbation calculation gives the coefficients explicitly.

For radiative absorption (reaction (3*c*)), $Q = 317$ MeV for Λ production, and $Q = 245$ MeV for Σ production; we have to include the coupling to the anomalous magnetic moments of the proton μ_p and the hyperons μ_Y. Moreover, there is the unknown interaction $\Sigma^0 \rightleftarrows \Lambda^0 + \gamma$ coming in. We assume, as in a previous paper [12], that the $(\Sigma^0\Lambda^0\gamma)$ interaction is of the Pauli type and its strength is of the order of the anomalous magnetic moment of the hyperon. The anomalous magnetic moments of the Σ hyperons are strongly suggested to be positive from the observed mass differences of the Σ triplet [13]. We also assume that $\mu_{\Lambda^0} = \mu_{\Sigma^0}$ in hyperon magnetons. The positive magnetic moments μ_{Λ^0} and μ_{Σ^0} interfere destructively with the contribution from μ_p. This is to be contrasted with the radiative absorption of slow negative pions in hydrogen, where μ_n and μ_p interfere constructively. For a numerical estimate, we used the values [13]:

$$\mu_{\Lambda^0} = 1.212\,, \qquad \mu_{\Sigma^0} = 1.138\,, \qquad \mu_p = 1.793$$

in nuclear magnetons.

[12] A. FUJII and R. E. MARSHAK: *Phys. Rev.*, **107**, 570 (1957).
[13] R. E. MARSHAK, S. OKUBO and E. C. G. SUDARSHAN: *Phys. Rev.*, **106**, 599 (1957).

The bi-mesonic absorption (reaction (3*d*)) is featured by the small Q-value (38 MeV). In our perturbation calculation, one diagram gives a large contribution, presumably reflecting the usual type of baryon pair effect associated with the pseudoscalar pion; hence, the numerical values cannot be taken too seriously.

The charge-exchange kaon absorption (reaction (3*a*)) takes place if the (K^-K^0) mass difference is greater than the neutron-proton mass difference of 1.29 MeV. If the mass difference of the charged and neutral kaons is entirely of electromagnetic origin, it is about $\frac{1}{3}$ of the mass difference between the charged and neutral pions ([14]). This gives 1.5 MeV, which makes the reaction slightly exothermic ($Q = 0.21$ MeV). The strangeness conservation allows only open diagrams like (*a*) and (*c*) of Fig. 1 and the weight factors in isotopic spin space make the contributions of these two diagrams almost cancel each other. For the (K^0n) mode, if we neglect a diagram contributed by the intermediate production of Λ^0, the transition rate becomes $(g_k^2/4\pi)^2 \cdot 2.07 \cdot 10^{16}$ s^{-1}, $(g_\kappa^2/4\pi)^2 \cdot 1.80 \cdot 10^{14}$ s^{-1} for the (S, 1*s*) and (PS, 1*s*) cases respectively.

The transition rates are summarized in Table III. We have not taken the trouble to compute the transition rates for the ($\Lambda^0\pi^0\pi^0$) channel which presumably are comparable to those for the ($\Lambda^0\pi^+\pi^-$) channel.

TABLE III. – *Transition rates in* s^{-1}.

Process		(S, 1*s*)	(PS, 1*s*)	(S, 2*p*)	(PS, 2*p*)
($\Lambda^0\gamma$)	$(g_k^2/4\pi)$ ×	$8.10 \cdot 10^{13}$	$4.69 \cdot 10^{14}$	$2.45 \cdot 10^{8}$	$1.29 \cdot 10^{8}$
($\Sigma^0\gamma$)	$(g_k^2/4\pi)$ ×	$7.08 \cdot 10^{13}$	$3.05 \cdot 10^{14}$	$3.89 \cdot 10^{8}$	$5.77 \cdot 10^{7}$
($\Lambda^0\pi^+\pi^-$)	$(g_k^2/4\pi)(g_\pi^2/4\pi)^2$ ×	$1.68 \cdot 10^{13}$	$5.21 \cdot 10^{12}$	$9.61 \cdot 10^{6}$	$4.33 \cdot 10^{7}$
(nK^0)	$(g_k^2/4\pi)^2$ ×	$1.11 \cdot 10^{14}$	$1.42 \cdot 10^{11}$	0	0

Each of these modes, apart from the questionable bi-mesonic absorption mode, is slow compared to the mesonic absorption mode. This could have been anticipated, of course; however, some of these slow reactions will be relatively more important in deuterium and the numerical estimates for hydrogen will be useful for purposes of comparison.

4. – Kaon absorption in deuterium.

The slow negative kaon reactions in deuterium give rise to three and four final particles except the non-mesonic absorption (4*a*). The maximum (total)

([14]) E. C. G. SUDARSHAN: *Ph. D. Thesis* (University of Rochester, 1957).

energy of the emitted pions or photons in the reactions (4*b*), (4*c*), (4*d*) are summarized below.

TABLE IV. – *Maximum total energy of the emitted quantum.*

Process	Λ production	Σ production
$K^- + d \to Y + N + \pi$	298 MeV	235 MeV
$K^- + d \to Y + N + \gamma$	291 MeV	225 MeV
$K^- + d \to \Lambda + N + 2\pi$	180 MeV	

4·1. *Mesonic absorption in deuterium.* – The matrix element for reaction (4*b*) is given by

$$\langle M \rangle = \int \Psi_f^*(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})\, O \Psi_i(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 ,$$

where Ψ_i and Ψ_f are the initial and final states of the two-baryon system. O is the transition operator

$$O = \frac{1}{\sqrt{2\omega_k}} \cdot \frac{\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}_1]}{\sqrt{V}}\, T(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}, \boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^1) \varphi_{n\cdot}(\boldsymbol{r}_1 - \boldsymbol{R}) , \tag{12}$$

where $\boldsymbol{R}$ is the co-ordinate of the center of mass of the deuteron and the process is supposed to take place on nucleon 1, $\boldsymbol{p}$ is the relative momentum of the final two-baryon system, φ_{nl} and $\nabla\cdot\varphi_{nl}$ are replaced by their values evaluated at the origin (center of mass of the deuteron) and removed from the integral. If the hyperon-nucleon force in the final state is neglected (see below for the justification of this approximation), the relative motion is represented by a plane wave with momentum $\boldsymbol{p}$. We use the Hulthen wave function for the deuteron. The matrix element becomes

$$\langle M \rangle = \frac{1}{\sqrt{2m}} \cdot \frac{1}{\sqrt{2\omega_k}} \frac{N_0}{\sqrt{V}} \langle \chi_f | \int \exp\left[-i\left(\boldsymbol{p} + \frac{M_0}{M + M_0}\boldsymbol{k}\right)\cdot\boldsymbol{r}\right] \cdot (T\cdot\varphi_{nl})_{\boldsymbol{r}_1} \mathfrak{H}(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}\, | \chi_i \rangle , \tag{13}$$

where

$$\mathfrak{H}(\boldsymbol{r}) = \frac{\exp[-\gamma r] - \exp[-\beta r]}{r}, \quad N_0 = \sqrt{\frac{\beta\gamma(\beta+\gamma)}{2\pi(\beta-\gamma)^2}}, \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2,$$

and χ_f, χ_i are the final and initial spin wave functions.

The general form of T can be derived by invariance arguments similar to

those for the hydrogen case: it is pseudoscalar for a scalar kaon and scalar for a pseudoscalar kaon, namely ([15]):

$$(14a) \qquad T(\mathrm{S}) = f_1(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + f_2(\boldsymbol{\sigma}\cdot\boldsymbol{k}) + f_3(\boldsymbol{\sigma}\cdot\boldsymbol{q}) ,$$

$$(14b) \qquad T(\mathrm{PS}) = g_1 + g_2\boldsymbol{p}^2 + g_3\,\boldsymbol{k}^2 + g_4(\boldsymbol{p}\cdot\boldsymbol{k}) + g_5(\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{k}) + \\ + g_6(\boldsymbol{p}\cdot\boldsymbol{q}) + g_7(\boldsymbol{k}\cdot\boldsymbol{q}) + g_8(\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{q}) + g_9(\boldsymbol{\sigma}\cdot\boldsymbol{k}\times\boldsymbol{q}) .$$

The terms independent of $\boldsymbol{q}$ are effective for the absorption from the $1s$ orbital and the terms linear in $\boldsymbol{q}$ are responsible for the absorption from the $2p$ orbital.

The probability of emitting a pion of momentum $k = |\boldsymbol{k}|$ is given by

$$(15) \qquad w(k) = \frac{g_{\mathrm{K}}^2}{4\pi}\cdot\frac{g_\pi^2}{4\pi}\cdot\tau\cdot\mathfrak{S}(k) ,$$

where τ is a characteristic reciprocal time,

$$\tau_{1s} = \frac{2}{\pi}\cdot\frac{M}{m}\cdot 2\pi N_0^2\cdot\frac{4}{a_0^3}\cdot M_0^{-3} = \begin{cases} 1.85\cdot 10^{16}\ \mathrm{s}^{-1} & (\Lambda) , \\ 1.97\cdot 10^{16}\ \mathrm{s}^{-1} & (\Sigma) , \end{cases}$$

$$\tau_{2p} = \frac{2}{\pi}\cdot\frac{M}{m}\cdot 2\pi N_0^2\cdot\frac{1}{24a_0^5}\cdot M_0^{-5} = \begin{cases} 1.78\cdot 10^{9}\ \mathrm{s}^{-1} & (\Lambda) , \\ 1.89\cdot 10^{9}\ \mathrm{s}^{-1} & (\Sigma) . \end{cases}$$

The momentum spectrum $\mathfrak{S}(k)$ reads

$$(16a) \qquad \mathfrak{S}(k;\mathrm{S},1s) = [\,|f_1|^2k^2 + |f_2|^2p^2]\cdot\frac{kI_1}{\omega_k} + 2\ \mathrm{Re}\,(f_1^*f_2)\cdot\frac{kI_2}{\omega_k} ,$$

$$(16b) \qquad \mathfrak{S}(k;\mathrm{PS},1s) = [\,|g_1|^2 + 2\ \mathrm{Re}\,(g_1^*g_2)p^2 + 2\ \mathrm{Re}\,(g_1^*g_3)k^2]\cdot\frac{kI_1}{\omega_k} + \\ + 2\ \mathrm{Re}\,(g_1^*g_4)\cdot\frac{kI_2}{\omega_k} ,$$

$$(16c) \qquad \mathfrak{S}(k;\mathrm{S},2p) = 3\,|f_3|^2\cdot\frac{kI_1}{\omega_k} ,$$

$$(16d) \qquad \mathfrak{S}(k;\mathrm{PS},2p) = [(|g_6|^2 + 2\,|g_8|^2)p^2 + (|g_7|^2 + 2\,|g_9|^2)k^2]\cdot\frac{kI_1}{\omega_k} + \\ + [2\ \mathrm{Re}\,(g_6^*g_7) + 4\ \mathrm{Re}\,(g_8^*g_9)]\cdot\frac{kI_2}{\omega_k} ,$$

([15]) This general form of the transition operator has been given already by PAIS and TREIMAN, ref. ([5]). They did not evaluate the coefficients, but some of their assumptions, e.g. $f_1 < f_2$, $g_5 < g_1$, $g_8 < g_9$, are confirmed by the present computation.

where p is a function of k through

$$p = \sqrt{\frac{2MM_0}{M+M_0}\left[M_0 + m - M - \frac{k^2}{2(M+M_0)} - \sqrt{k^2+\mu^2}\right]},$$

and

$$I_1 = J_1 , \tag{17a}$$

$$I_2 = \frac{M+M_0}{2MM_0}\left[J_2 - \left(p^2 + \left(\frac{M_0}{M+M_0}\,k\right)^2\right)J_1\right], \tag{17b}$$

the J's being given by:

$$J_1 = \int_{|P-(M_0/(M+M_0))k|}^{P+(M_0/(M+M_0))k} \left[\frac{1}{x^2+\gamma^2} - \frac{1}{x^2+\beta^2}\right]^2 x\,\mathrm{d}x\,, \tag{18a}$$

$$= \left[\frac{-1}{2}\left(\frac{1}{x^2+\gamma^2} + \frac{1}{x^2+\beta^2}\right) - \frac{1}{\beta^2-\gamma^2}\log\left|\frac{x^2+\gamma^2}{x^2+\beta^2}\right|\right]_{|P-(M_0/(M+M_0))k|}^{P+(M_0/(M+M_0))k}.$$

$$J_2 = \int_{|P-(M_0/(M+M_0))k|}^{P+(M_0/(M+M_0))k} \left[\frac{1}{x^2+\gamma^2} - \frac{1}{x^2+\beta^2}\right]^2 \cdot x^3\,\mathrm{d}x\,, \tag{18b}$$

$$= \left[\frac{1}{2}\left(\frac{\gamma^2}{x^2+\gamma^2} + \frac{\beta^2}{x^2+\beta^2}\right) + \frac{\beta^2+\gamma^2}{2(\beta^2-\gamma^2)}\log\left|\frac{x^2+\gamma^2}{x^2+\beta^2}\right|\right]_{|P-(M_0/(M+M_0))k|}^{P+(M_0/(M+M_0))k}.$$

We find that I_2 is much smaller than I_1 for all values of k, actually at most a few per cent, and negative.

The total transition rate is obtained by numerical integration over the momentum spectrum. The momentum, or rather energy spectrum of the emitted pion, is highly peaked at the high energy end, the half-width being of the order of $(10 \div 15)$ MeV. The spectral shape for the (S, $2p$) case (i.e. I_1) is drawn in Fig. 2 using an arbitrary scale. There is not much difference for the other cases insofar as the pion energy spectrum is concerned.

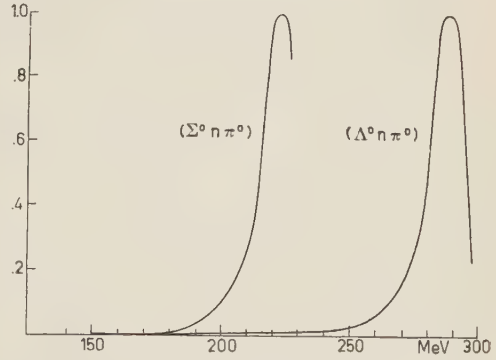

Fig. 2. – Energy spectrum of the emitted pions, (S, $2p$). The ordinate is on an arbitrary scale, normalized at the peak value. The abscissa is total pion energy.

The coefficients f and g are given in Tables V and VI.

We have only listed the re-

sults for three channels because the results for the other channels follow from these three (see below).

TABLE V. – *Coefficient f.*

	$(\Lambda^0 n\pi^0)$	$(\Sigma^0 n\pi^0)$	$(\Sigma^0 p\pi^-)$
f_1	-0.134	-0.251	-0.273
f_2	0.407	1.113	1.174
f_3	0.122	0.103	0.153

TABLE VI. – *Coefficient g.*

	$(\Lambda^0 n\pi^0)$	$(\Sigma^0 n\pi^0)$	$(\Sigma^0 p\pi^-)$
g_1	-0.960	-0.943	-1.319
g_2	0.887	0.717	1.262
g_3	0.546	0.509	0.804
g_4	0.127	0.096	0.262
g_5	$-0.496i$	$-0.487i$	$-0.774i$
g_6	-2.192	-1.888	-3.142
g_7	-0.818	-0.234	-0.779
g_8	$0.247i$	$0.153i$	$0.261i$
g_9	$-1.239i$	$-1.120i$	$-1.917i$

For the (S, 1s) case, $f_2 > f_1$, i.e. the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{k})$ is weighted more heavily than $(\boldsymbol{\sigma}\cdot\boldsymbol{p})$; that is to say, emission of a p-wave pion is more probable than that of an s-wave pion with the hyperon-nucleon coming out in a relative p-state. Since the spectrum is peaked for high k, p is small. For the (PS, 1s) case, s-wave pion emission is expected. For the (PS, 2p) case with g_6 and g_9 dominant, the pion is emitted both in s- and p-waves.

The total transition rates are given in Table VII.

TABLE VII. – *Total transition rates in* s^{-1}.

Process		(S, 1s)	(PS, 1s)	(S, 2p)	(PS, 2p)
$(\Lambda^0 n\pi^0)$	$(g_k^2/4\pi)(g/_\pi^2 4\pi)\times$	$9.75\cdot10^{14}$	$5.31\cdot10^{16}$	$2.80\cdot10^{8}$	$1.94\cdot10^{9}$
$(\Sigma^0 n\pi^0)$	» $\times$	$2.17\cdot10^{15}$	$3.45\cdot10^{16}$	$1.77\cdot10^{8}$	$6.04\cdot10^{8}$
$(\Sigma^0 p\pi^-)$	» $\times$	$2.28\cdot10^{15}$	$6.70\cdot10^{16}$	$2.81\cdot10^{8}$	$9.88\cdot10^{8}$

Charge independence implies the relations:

$$w(\Lambda^0 p\pi^-) = 2w(\Lambda^0 n\pi^0)\,, \qquad w(\Sigma^- p\pi^0) = w(\Sigma^0 p\pi^-)$$

and also in the lowest order perturbation, it turns out that

$$w(\Sigma^+ n\pi^-) \cong 0 ,$$

$$w(\Sigma^- n\pi^+) \cong 4w(\Sigma^0 n\pi^0) .$$

Thus, from charge independence, we can say rigorously that:

Transition rate summed over all Λ-modes

$$= 3w(\Lambda^0 n\pi^0)$$

Transition rate summed over all Σ-modes

$$= 3[w(\Sigma^0 n\pi^0) + w(\Sigma^0 p\pi^-)].$$

The optical transition rate reads

$$w_{2p\to 1s} = 4.80 \cdot 10^{11}\ s^{-1} .$$

Again, using the rough estimates of the kaon coupling constant quoted in Sect. 3, we find that for a scalar kaon the optical transition dominates by about a factor 10 over the absorption from the $2p$ orbital, while for a pseudoscalar kaon both are comparable or the $2p$ absorption may even be larger. The increase of the number of the channels in deuterium enhances the $2p$ absorption compared to hydrogen.

The hyperon-nucleon force acting in the final state will distort the assumed plane wave function and presumably sharpen up the energy spectrum. Since the spectrum is already fairly sharp and the hyperon-nuclear force is not known too well, we have not calculated this effect which, in any case, should not change the total transition probability ([1]).

We should like to comment on several other aspects of the mesonic absorption process in deuterium. We might expect to be able to distinguish between the $2p$ and $1s$ absorption by investigating the angular distribution of the outgoing pions. However, this appears to be extremely difficult for the following reason. Suppose that the internal motion of the deuteron is neglected; then the absorption of the kaon takes place at rest. In the framework of the impulse approximation, we have a two body reaction and the hyperon and pion move away in opposite directions, namely $\boldsymbol{p}$ is anti-parallel to $\boldsymbol{k}$. The angular distribution of the pions with respect to the angle θ between $\boldsymbol{p}$ and $\boldsymbol{k}$ is thus concentrated around $\theta = \pi$. The internal motion of the nucleons within the deuteron spreads out this angular distribution so that the

« kinematical » angular distribution (using the Hulthen wave function) is the sharply peaked function

$$(19) \qquad \left[\frac{1}{(\boldsymbol{p}+M_0/(M+M_0)\boldsymbol{k})^2+\gamma^2}-\frac{1}{(\boldsymbol{p}+M_0/(M+M_0)\boldsymbol{k})^2+\beta^2}\right]^2,$$

whose width is of the order of 10 degrees or so. Hence, the details of the dynamics are obscured. If we go beyond the impulse approximation, the « kinematical » angular distribution will presumably be less peaked.

In principle, the polarization of the hyperon resulting from the mesonic absorption of kaons in deuterium should depend on the nature of the kaon interaction. If there is any tendency to produce the hyperon with a preferential spin direction (with respect to the production plane), it is possible to detect this effect by looking at the asymmetry of the subsequent decay process $Y \to N+\pi$ ([16]). Defining the direction of the spin parallel to $\boldsymbol{p}\times\boldsymbol{k}$, (we recall that $\boldsymbol{k}$ is the pion momentum, $\boldsymbol{q}$ is the relative momentum of the hyperon-nucleon system), we find the degree of polarization

$$P(\theta)=\frac{|\langle\uparrow\rangle|^2-|\langle\downarrow\rangle|^2}{|\langle\uparrow\rangle|^2+|\langle\downarrow\rangle|^2},\quad \theta=\sphericalangle(\boldsymbol{p},\boldsymbol{k}),$$

where $\langle\uparrow\rangle$ and $\langle\downarrow\rangle$ denote the spin up and down amplitudes of the final hyperon, to be

$$(20a) \qquad P(\theta;\mathrm{S},1s) = \frac{2\sin\theta\ \mathrm{Re}\,(if_1^*f_2)pk}{|f_1|^2p^2+|f_2|^2k^2+2\cos\theta\ \mathrm{Re}\,(f_1^*f_2)pk},$$

$$(20b) \qquad P(\theta,\mathrm{PS},1s) = \frac{2\sin\theta\ \mathrm{Re}\,(g_1^*g_5)pk}{|g_1|^2+2\ \mathrm{Re}\,(g_1^*g_2)p^2+2\ \mathrm{Re}\,(g_1^*g_3)k^2+2\cos\theta\ \mathrm{Re}\,(g_1^*g_4)pk},$$

$$(20c) \qquad P(\theta;\mathrm{S},2p) = 0,$$

$$(20d) \qquad P(\theta;\mathrm{PS},2p) =$$

$$= \frac{-2\sin\theta\ \mathrm{Re}\,(ig_8^*g_9+g_6^*g_9-g_7^*g_8)pk}{(|g_6|^2+2|g_8|^2)p^2+(|g_7|^2+2|g_9|^2)k^2+2\cos\theta\ \mathrm{Re}(g_6^*g_7+2g_8^*g_9)pk}.$$

We see that the polarization arises from the interference of the production of the hyperons in relative s and p-states with respect to the residual nucleons.

([16]) T. D. Lee, J. Steinberger, G. Feinberg, P. K. Kabir and C. N. Yang: *Phys. Rev.*, **106**, 1367 (1957).

Unfortunately, in our lowest order perturbation calculations (« Born approximation »), all numerators vanish, predicting zero polarization. It is necessary to do more refined calculations to obtain non-vanishing polarizations.

Finally, we remark on the multiple (pion) scattering effects on the mesonic absorption of kaons in deuterium. The mesonic absorption may be interpreted as the superposition of two fundamental processes:

$$\text{primary} \qquad K^- + N_1 \to Y + \pi'$$

$$\text{secondary} \qquad \pi' + N_2 \to N_2 + \pi$$

shown schematically in Fig. 3. The secondary process is a virtual pion-nucleon scattering. Assuming that the $(\frac{3}{2}, \frac{3}{2})$ resonance for pion-nucleon scattering is dominant, we note that:

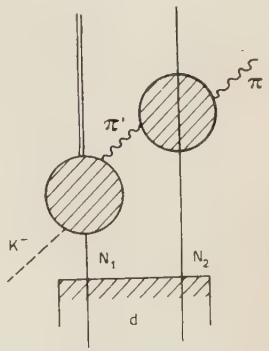

Fig. 3. – Schematic diagram of the multiple scattering.

a) For Λ-production there is no appreciable secondary effect since the system $(\Lambda N\pi)$ must have total isotopic spin $\frac{1}{2}$ (the initial K^-d system does) so that the sub-system $(N\pi)$ cannot have isotopic spin $\frac{3}{2}$.

b) In the static limit of the primary reaction, if π' is produced as a *p*-wave with respect to Y, then its parity is plus. The initial kaon-nucleon system must have positive parity too: i.e. if the kaon is scalar it is in an *s*-orbit and if it is pseudoscalar it is in a *p*-orbit. Since the neighboring nucleon is in an *s*-state with respect to the absorbing nucleon, the π' is also relatively in a *p*-state with respect to the neighboring nucleon. It follows that the effect of the secondary scattering will be appreciable only for Σ-production in the (S, 1*s*) and (PS, 2*p*) cases. In fact, if the primary reaction occurs as a real process the pion energy is about 100 MeV even in the static limit, which is certainly off resonance. Hence the effect of the secondary scattering will not be too large.

We estimated the change of the branching ratios due to this secondary scattering for the (S, 1*s*) case using a « static model » ([17]). The branching ratio of the primary process is adjusted so as to reproduce the experimental branching ratios in hydrogen. The unknown amplitude for the absorption by the neutron is computed with the aid of isotopic spin conservation. The results are shown in Table VIII, where the sets I and II refer to the two solutions for

([17]) H. Miyazawa: *Phys. Rev.*, **104**, 1741 (1956).

the relative phase of the production amplitude L

$$L_1 = L(K^- + p \to \Sigma^+ + \pi^-)\,, \qquad L_2 = L(K^- + p \to \Sigma^- + \pi^+)\,,$$

Set I $$\frac{L_2}{L_1} = \sqrt{2}\exp[i\varphi]\,, \qquad \varphi = \quad 69.2^0\,,$$

Set II $$\frac{L_2}{L_1} = \sqrt{2}\exp[i\varphi]\,, \qquad \varphi = -69.2^0\,.$$

TABLE VIII. – *Branching ratios.*

Process	No correction	Set I	Set II
$(\Sigma^+ n\pi^-)$	1	0.61	0.61
$(\Sigma^0 n\pi^0)$	1	1.04	1.34
$(\Sigma^- n\pi^+)$	2	1.91	2.21
$(\Sigma^0 p\pi^-) = (\Sigma^- p\pi^0)$	1	0.81	1.11

It is seen that the multiple scattering effect is no more than a factor of two.

4·2. *Other modes of absorption in deuterium.* – Among the three radiative modes $(\Lambda^0 n\gamma)$, $(\Sigma^0 n\gamma)$, $(\Sigma^- p\gamma)$ (reaction (4*c*)), the first two have the same character as discussed in Sect. 3.2. Assuming again that Σ^- possesses a positive anomalous magnetic moment, it contributes additively to the negative magnetic moment of the neutron so that the $(\Sigma^- p\gamma)$ process is enhanced by about a factor 20 compared to the other radiative processes for the (S, 1*s*) case. At the same time, the kaon current contribution for the (S, 2*p*) case is large for the $(\Sigma^- p\gamma)$ process but it is partly cancelled for the other two processes. Hence a rather large transition rate is expected for the $(\Sigma^- p\gamma)$ mode for a scalar kaon.

The bi-mesonic absorption (reaction (4*d*)) is a four body problem for deuterium and we merely estimated an upper bound on the transition rate. It should again be noted that the large influence of baryon pairs in this reaction probably leads to a gross overestimate.

The non-mesonic absorption (reaction (4*a*)) would, at first sight, be expected to compete significantly with mesonic absorption. However, the kinetic energy acquired by the two final baryons is so high (315 MeV for the Λ modes, 245 MeV for the Σ modes) that the overlap of the final wave function with the deuteron wave function is very small. In the case of mesonic absorption the pion takes away most of the available kinetic energy so that there is a substantial overlap of the deuteron and final two-baryon wave functions. This results in dominance of the mesonic absorption reactions in most instances.

The charge exchange kaon reaction in deuterium requires the mass difference of charged and neutral kaons to be greater than 3.5 MeV. Since the

theoretical prediction is 1.5 MeV for the kaon mass difference, the reaction should not go.

The calculated transition rates are summarized in Table IX.

TABLE IX. – *Transition rates in* s^{-1}.

Process		(S, 1*s*)	(PS, 1*s*)	(S, 2*p*)	(PS, 2*p*)
$(\Sigma^- p\gamma)$	$(g_k^2/4\pi)\times$	$7.66\cdot10^{15}$	$1.99\cdot10^{14}$	$6.36\cdot10^{9}$	$6.92\cdot10^{7}$
$(\Lambda^0 n)$	» $\times$	$4.28\cdot10^{15}$	$2.33\cdot10^{14}$	$6.75\cdot10^{6}$	$8.74\cdot10^{7}$
$(\Sigma^0 n)$	» $\times$	$8.24\cdot10^{15}$	$3.26\cdot10^{14}$	$9.41\cdot10^{6}$	$1.62\cdot10^{8}$
$(\Lambda^0 n\pi^+\pi^-)$	$(g_k^2/4\pi)(g_\pi^2/4\pi)^2\times$	$1.81\cdot10^{13}$	$8.14\cdot10^{13}$	$1.34\cdot10^{8}$	$6.96\cdot10^{7}$

Comparing the above values to the mesonic absorption rates (Table VII), we note that these «rare» modes are quite small for $2p$-absorption for both scalar and pseudoscalar kaons. The ratios of the «rare» to mesonic modes show some differences between a scalar and pseudoscalar kaon for $1s$ absorption. For a scalar kaon, the dominant «rare» process is non-mesonic and radiative, and the «rare» to mesonic ratio is of the order of 10^{-1}. For a pseudoscalar kaon, this ratio is of the order of 10^{-3}. The reasons for this difference may be understood as follows. If the kaon is scalar, it is readily absorbed by the baryon in a positive to positive energy transition; on the other hand, the emission of a pseudoscalar pion is inhibited. If the kaon is pseudoscalar, the simple absorption corresponding to the positive energy to positive energy is inhibited, but if it is accompanied by the emission of the pion, it proceeds readily (through the creation of virtual baryon pairs).

5. – Summary.

The transition rates for the competing reactions resulting from the absorption of slow negative kaons from the $1s$ and $2p$ orbitals in hydrogen and deuterium have been calculated on the basis of a charge independent interaction Hamiltonian with universal pion-baryon coupling and universal kaon-baryon coupling in the lowest order perturbation theory assuming that the hyperons and kaons have spin $\frac{1}{2}^+$ and $0^\pm$ respectively.

If the kaon is scalar, the absorption from the $2p$ orbital is about a factor 10 smaller than the optical transition rate in both hydrogen and deuterium. From the $1s$ orbital, the absorption takes place quickly and gives rise chiefly to the mesonic mode in hydrogen. In deuterium, the absorption from the $1s$ orbital favours the mesonic mode too, but the non-mesonic and the radiative absorption modes should compete on the order of 10%.

If the kaon is pseudoscalar, the absorption from the $2p$ orbital and the optical transition rate are comparable. The reaction mode is mainly mesonic both in hydrogen and deuterium, from both the $1s$ and $2p$ orbitals.

The predicted branching ratios for the mesonic reactions in hydrogen agree with some features of the experimental results (e.g. the dominance of the $(\Sigma^-\pi^+)$ mode) and differ from other s (e.g. the occurrence of the $(\Sigma^+\pi^-)$ mode with reasonable frequency). The theoretical suppression of the $(\Sigma^+\pi^-)$ mode follows from the assumption of global symmetry and does not depend on our use of lowest order perturbation theory. If the experimental results persist, it will be necessary to reconsider the hypothesis of global symmetry.

For the deuterium reactions, the effects of multiple scattering of the pion and of the hyperon-nucleon force in the final state are not very large, so that the qualitative predictions of the impulse approximation are unaltered.

* * *

It is a pleasure to thank Dr. C. J. GOEBEL and Mr. B. SAKITA for several useful discussions.

RIASSUNTO (*)

Sulla base di un semplice modello si sono calcolate le probabilità di transizione per le reazioni concorrenti risultanti dall'assorbimento in idrogeno di mesoni K negativi lenti. I calcoli sono stati fatti per gli accoppiamenti scalare e pseudoscalare del mesone K da parte degli orbitali $1s$ e $2p$ dei differenti atomi K^- mesici. L'assorbimento da parte dello stato $2p$ è favorito rispetto al caso del pione data la minor dimensione dell'orbita. Quest'effetto tende a ridurre la possibilità di distinguere tra i caratteri scalare e pseudoscalare dell'accoppiamento dei mesoni K nelle interazioni forti. L'emissione di un singolo pione è favorita sia nel deuterio che nell'idrogeno. Si traggono conclusioni qualitative concernenti l'importanza relativa degli altri modi di assorbimento dei mesoni K sia in idrogeno che in deuterio.

(*) *Traduzione a cura della Redazione.*

Interactions and Decays of K⁻ Mesons.

II. - Analysis of the Decays.

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(ricevuto il 12 Febbraio 1958)

Summary. — In a systematic study of about 1600 K^--mesons in nuclear emulsions, we have observed 2 τ^- and 50 events where the primary K^- disappears in flight, giving rise to nothing but one «light track». 12 of these events had an associated blob at the K^- disappearance point and 38 had no such an associated blob. It is shown that most of the 38 events without an associated blob are examples of K^- decay in flight. Detailed analysis of these events gives the following results: *a*) The $K^-_{\mu 2}$ and the $K^-_{\pi 2}$ decay modes should be considered as established. The mean C.M.S. kinetic energies of the secondaries observed were: $K_{\mu 2}$; $T_\mu^{CM} = (153 \pm 4)$ MeV, $K_{\pi 2}$; $T_\pi^{CM} = (109.5 \pm 3)$ MeV. *b*) The relative abundance of the $K_{\mu 2}$ is about 50%. *c*) The mean K^- lifetime is $\tau_{K^-} = (1.60 \pm 0.3) \cdot 10^{-8}$ s.

1. – Introduction.

The various decay modes of the positive K-meson have been studied recently in detail by many authors (1). Thus, in particular, it has been established that 57% of all positive K-decays are $K^+_{\mu 2}$. It is naturally interesting

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(1) A compilation of the experimental material available as of March 1957 was published recently by C. Franzinetti and G. Morpurgo: *Suppl. Nuovo Cimento*, **6**, 469 (1957). See also: G. Alexander, R. H. W. Johnston and C. O'Ceallaigh: *Nuovo Cimento*, **6**, 478 (1957).

to determine the relative abundance of the various K -decay modes and compare it with the relative abundance of the K^+-decay modes. If indeed K^+ and K^- are particle and anti-particle, it should be expected that the relative abundance of the various decay modes should be the same.

Contrary to K^+-mesons which decay upon coming to rest, K^--mesons are absorbed and produce « super-sigma stars ». Thus, the study of the K -decay modes has to be performed on decays in flight which are quite rare and which are not so easily analysable. Some examples of K^--decays have been published ([2]) (see also ref. ([1])) but no systematic and detailed study of the problem has so far been published.

The purpose of this work is to establish the K -decay modes, to determine the relative abundance and the kinetic energy carried in the center of mass system by the charged secondaries in each decay mode, and to determine the lifetime of the negative K-mesons.

The experimental procedure and the decay selection criterion are discussed in Sect. **2**. The results obtained are presented in Sect. **3**.

2. - Experimental procedure and the decay selection criterion.

In a systematic study of about 1600 K^--mesons in nuclear emulsions ([3]), 420 K^- interactions in flight, 2 τ^--decays and 50 events where the primary K disappears in flight and only one « light track » emerges from the K disappearance point were observed. Among these 50 events (hereafter referred to as decay-like events) 12 events had an associated blob at the K^- disappearance point an in 38 events no associated blob was observed. In principle each one of the 50 decay-like events could be either a decay or an interaction in flight of the K with (probably) a peripheral neutron giving only one thin track. However, examination of K^- elastic scatterings in the same emulsions indicated that in only 5% of the scatterings a blob could be associated with the deflection point. Thus, probably most of the decay-like events with an associated blob are examples of K^- stars in flight. These 12 events will be discussed in detail in a forthcoming paper ([4]).

We shall presently show that about 90% of the decay-like events with

([2]) J. HORNBOSTEL and G. T. ZORN: *Phys. Rev.*, **109**, 165 (1958).

([3]) E. LOHRMANN, M. NIKOLIĆ, M. SCHNEEBERGER, P. WALOSCHEK and H. WINZELER: *Nuovo Cimento*, **7**, 163 (1958). General results and details of the experiments are given there.

([4]) Y. EISENBERG, W. KOCH, E. LOHRMANN, M. NIKOLIĆ, M. SCHNEEBERGER and H. WINZELER: *Interactions and Decays of* K^-*-mesons*, part III, to be published soon.

no associated blob are, indeed, examples of K^- decay in flight. Thus, in accepting the blob criterion strictly, that is, in assuming that the decay-like events without blob are all decays and that the decay-like events with blob are all stars in flight, no great error is made.

The reasons which led us to believe that most of the decay-like events without blob are indeed decays, are as follows:

a) Investigation of K^- stars in flight in which only one gray particle was emitted, showed that only 30% have no associated blob. We observed 12 decay-like events with blob. Thus we should expect to have about 4 stars among the decay-like events with no blob. This number is most probably not underestimated since a blob is harder to recognize in an event having a gray secondary than in an event having a light secondary, whereas the recoil energy is probably not very different in both events.

b) An upper limit for the number of stars present among the decay-like events without blob could be obtained by the following argument. Study of the decay-like stars with blob gave the result that only about 30% of the secondaries are emitted in such an angle and with such an energy as to be compatible with one of the common K-decay modes ($K_{\mu 2}$ or $K_{\pi 2}$). Thus 70% of decay-like stars in flight do not resemble K-decays. Among the well analysable decay-like events without blob (12 cases) having a flat secondary, which is a random sample of all decay-like events without blob, we did not find a single event which was not compatible with one of the common K-decay modes. Considering the statistical significance of our data, we conclude that among all 38 decay-like events with no associated blob there are at most 6 stars.

c) In all the above well analysable 12 events without blob we obtained excellent agreement with one of the known K^+ decay modes. Furthermore, the mean Q-values obtained also agree quite well with the corresponding K^+ Q-values, as should be expected since K^+ and K^- have approximately the same mass ([1]). It should be noted that in 7 out of the 12 cases the possibility of an interaction is completely ruled out even if we take into account the extremes of the pion energy spread due to a reasonable Fermi momentum of the target nucleon (this pion energy spread is presented graphically in ref. ([4])). If among the rest of the events there were many stars, it would be unlikely that the secondary energy, which should have quite a big Fermi spread, should always agree with the energy predicted by assuming one of the established K-decay modes. Thus probably most of the events are true decays.

d) The K^- lifetime obtained by assuming that most of the decay-like events without blob are decays (for details, see Sect. 3 of this work) is in

reasonable agreement with other determinations of the K^- lifetime ([5]) and with the K^+ lifetime ([1]). It should be noted that our K^- lifetime is already quite long and that, if we assume that K^- and K^+ have exactly the same lifetime, we should see more than 38 decays. As was mentioned before, the observation of about 38 decays is still compatible with the K^+ lifetime, if we consider the statistical errors. To assume that many of the decay-like events without blob are stars would mean that the K lifetime is about twice or more longer than the K^+ lifetime.

Considering all the above arguments, it seems reasonable to assume that almost all of the decay-like events which do not have an associated blob are examples of a negative K-meson decay in flight. In the analysis presented in the next section this will be assumed.

3. – Results and discussion.

As was mentioned before, not all 38 events could be analysed due to the difficulties in scattering steep tracks. The energy, $p\beta$ and ionization of the secondaries are determined uniquely ([6]) by the energy of the primary K^- and the angle θ between the primary and secondary particles. The events which we were able to analyse could be classified into two main groups:

a) The scattering group. All events with a flat secondary, where a reliable determination of $p\beta$ was possible, were included in this group.

b) The ionization group, including all events of predicted secondary particle ionization (for a $K_{\pi 2}$-decay mode) greater than 1.2 minimum and which could not be scattered. The actual measured ionization was compared with the predicted values and since for the same T_K and θ a muon is less ionizing than a pion, one could differentiate between a $K_{\mu 2}$ and a $K_{\pi 2}$.

The events which could not be classified in the above manner (a total of 16) were left out of the analysis. We think that the above selection method is practically unbiased and that the sample of 22 events admitted into the final analysis is approximately a random one.

Naturally more detailed information was obtained in class *a*) events (see Table I below); but some of the class *b*) secondaries stopped in the emulsion

([5]) *Padua-Venice Conference* (Sept. 1957), compilation of all laboratories results by G. GOLDHABER.

([6]) H. S. WHITE: UCRL Report No. 3514 (Sept. 1956).

TABLE I. – *The observed and calculated data for the secondaries in the 22 decay-like events without an associated blob.*

Event	Calculated Secondary Quantities: Assuming $K_{\mu 2}$			Assuming $K_{\pi 2}$			Measured Secondary Quantities			Most Probable Decay Mode	Remarks
	$p\beta$ $\left(\frac{\text{MeV}}{c}\right)$	Ionization g^*	Range (cm)	$p\beta$ $\left(\frac{\text{MeV}}{c}\right)$	Ionization g^*	Range (cm)	$p\beta$ $\left(\frac{\text{MeV}}{c}\right)$	Ionization g^*	Range (cm)		
Group (*a*)											
K_{270}	317	—	—	255	—	—	339 ± 42	—	—	$K_{\mu 2}$	
B_{470}	382	—	—	337	—	—	368 ± 38	—	—	$K_{\mu 2}$	
B_{420}	337	—	—	285	—	—	344 ± 44	—	—	$K_{\mu 2}$	
Ch_{91}	188			138	—	—	192 ± 25	—	—	$K_{\mu 2}$	
Lo_{124}	304	—	—	250	—	—	330 ± 39	—	—	$K_{\mu 2}$	
K_{226}	368	—	—	319	—	—	333 ± 33	—	—	$K_{\pi 2}$	
M_{13}	411	—	—	358	—	—	366 ± 70	—	—	$K_{\pi 2}$	
Ch_{83}	357	—	—	305	—	—	306 ± 33	—	—	$K_{\pi 2}$	
B_{350}	341	—	—	286	—	—	298 ± 32	—	—	$K_{\pi 2}$	Secondary makes a star in flight: Calculated $p\beta$ for a $K_{\pi 3}$ decay: $p\beta \leqslant 217$ MeV/c.
I_{273}	390	—	—	340	—	—	192 ± 24		—	$K_{\pi 3}$ (1)	
Groups (*a*)+(*b*)											
Lo_{115}	157	1.0	—	113	1.24	—	158 ± 14	1.0	—	$K_{\mu 2}$	
B_{400}	—	—	6.20	—	—	1.80	—	—	1.76	$K_{\pi 2}$ (2)	Secondary stops and makes a σ2 star.
Group (*b*)											
K_{175}	—	—	6.10	—	—	2.40	—	—	5.80	$K_{\mu 2}$ (3)	Secondary stops and makes a σ0 star.
Ch_{135}	—	0.96		—	1.25	—	—	~ 1.0	—	$K_{\mu 2}$ (4)	
K_{278}	—	0.97	—	—	1.20	—	—	~ 1.0	—	$K_{\mu 2}$ (4)	
B_{185}	—	1.00	—	—	1.23	—	—	~ 1.07	—	$K_{\mu 2}$ (4)	
B_{13}	—	1.00	—		1.23	—	—	~ 1.0	—	$K_{\mu 2}$ (4)	
B_{411}	—	1.14	—	—	1.73	—	—	~ 1.1	—	$K_{\mu 2}$ (4)	
T_{51}	—	1.1	—	—	1.60	—	—	~ 1.0	—	$K_{\mu 2}$ (4)	
H_{123}	—	0.96	—	—	1.27	—	—	?	—	$K_{\pi 2}$ (5)	Secondary makes a star in flight.
K_{276}	—	—	12	—	—	5.2	—	—	4.3	? (6)	Secondary stops and makes a σ4 star.
B_{447}	—	—	14.8	—	—	6.5	—	—	4.1	? (6)	(*)

(*) Secondary stops and makes a σ0 star. Measured secondary mass: $M_{sec} = (267 \pm 20)\ m_e$.

(1) Secondary must be a pion since it makes a star in flight. The probability of this event to be a K^- interaction is only 10 %, since, in order to account for the observed π energy, very high Fermi momentum for the target nucleon has to be assumed.

(2) Secondary was flat so event belongs also to group (*a*).

(3) Direct mass determination of secondary particle indicates that it is a muon: $M_{sec} = (185 \pm 30)\ m_e$.

(4) Quite poor information is available in these events. $K_{\pi 2}$ and $K_{\pi 3}$ are excluded. So, these events could be $K_{\mu 2}$, $K_{\mu 3}$ or K_{e3}.

(5) Secondary is a pion since it makes a star in flight. Secondary is too short for a good g^* determination but it seems to be about minimum, so $K_{\pi 3}$ is excluded.

(6) These two cases are probably examples of K^- interactions since secondaries are certainly pions but they do not correspond to any of the known K_π decay modes. These events have been excluded from the analysis.

block. In these cases the nature of the secondary and its energy at the decay point could be determined exactly. In the rest of class *b*) events, however, one could exclude either the $K_{\mu 2}$ or $K_{\pi 2}$ decay mode which contain about 80% of all decays of the K^+.

The entire analysis was conducted under the assumption that the K^--meson does not have any decay modes different from those of the K^+-meson. The results are summarized in Table I. The calculated secondary quantities were obtained from the geometry of the event and the measured primary K^- energy. The decision whether any particular event is a $K_{\mu 2}$ or a $K_{\pi 2}$ was made by comparing the observed and calculated quantities and by taking into account the statistical error. In Fig. 1 we plot the number of events vs. the probability that each particular event is a pion, P_π, divided by the sum $P_\pi + P_\mu$. (Note that for infinitely good resolution $P_\pi/(P_\pi + P_\mu)$ should be either 0 or 1). By inspecting Fig. 1 it becomes evident that most of the events can be clearly separated. The decay mode was uniquely determined in 9 events out of the 22 analysed.

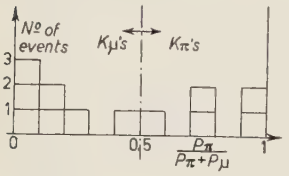

Fig. 1. – No. of events vs. the ratio $P_\pi/(P_\pi+P_\mu)$. P_π and P_μ are the probabilities that each event is a $K_{\pi 2}$ or $K_{\mu 2}$ respectively.

TABLE II. – *The relative abundance of the various* K_L^-*-meson decay modes (in percent).*

All Events		Scattering Group alone	
Decay Modes	Relative Abundance in %	Decay Modes	Relative Abundance in %
$K_{\mu 2}$ plus the K_{e3}'s and some of the $K_{\mu 3}$'s	65 ± 18	$K_{\mu 2}$	50 ± 20
$K_{\pi 2}$ plus some of the $K_{\mu 3}$'s and perhaps one K_{e3}	30 ± 12	$K_{\pi 2}$ plus the $K_{\mu 3}$'s and perhaps one K_{e3}	42 ± 19
$K_{\pi 3}$	5 ± 5	$K_{\pi 3}$	8 ± 8

Also, one τ^- out of a total of 39 decays should be included.

The observed energy divided by the calculated energy, in the laboratory system, and the kinetic energy in the K^- rest system, are plotted in

Fig. 2 and 3 respectively, for the 12 well analysed 2-body events. Again it should be emphasized, perhaps, that such an agreement between the observed and calculated energies is not expected if many of the decay-like events without an associated blob were interactions.

The weighted center of mass kinetic energies, on the basis of these 12 events are:

$$K^-_{\mu 2}: \qquad T_\mu^{CM} = (153 \pm 4)\ \text{MeV}\,,$$

$$K^-_{\pi 2}: \qquad T_\pi^{CM} = (109.5 \pm 3)\ \text{MeV}\,,$$

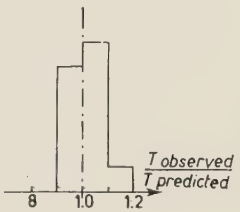

Fig. 2. – (T observed)/(T calculated) for the 12 well analysable 2-body events.

which are in agreement with the Q-values in the K^+ decay.

The relative abundance of the various K^--decay modes can be computed for the entire sample of 22 events or for the scattering group alone (this group by itself is also unbiased). The results, presented in Table II, indicate that the relative abundances of the various decay modes of the K^- are in agreement with those of the K^+.

On the basis of the present experiment, we can also determine the K^- lifetime. As was already discussed in Sect. 2, 6 stars could be present among the 38 decay-like events without blob and perhaps two decays among the decay-like events with blob (*). Also, as was discussed in part I ([3]), some decays could be among the K^- disappearance events with no visible secondary. Taking all of this into account and following the procedure of ref. ([3]), we obtain for the K^- lifetime:

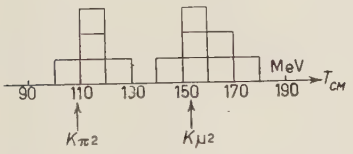

Fig. 3. – C.M. kinetic energies of the 12 well analysable $K_{\mu 2}$ and $K_{\pi 2}$ events.

$$\tau_{K^-} = (1.60 \pm 0.3)\cdot 10^{-8}\ \text{s}\,.$$

We have thus demonstrated the existence of the $K^-_{\mu 2}$ and the $K^-_{\pi 2}$ decay modes (events K 175 and B 400 are very good examples of $K_{\mu 2}$ and $K_{\pi 2}$ decays respectively; see Table I) and have also shown that the mean Q-values obtained in the above decays agree with those of the positive K-meson. Also, the K lifetime and the relative abundance of the various decay modes are consistent with the corresponding values of the K^+-meson. This is to be expected if indeed K^+ and K^- are particle and anti-particle.

(*) With these corrections the final number of decay-like stars becomes 16.

* * *

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RIASSUNTO (*)

In uno studio sistematico, fatto in emulsione nucleare, di circa 1 600 mesoni K^- abbiamo osservato 2 τ^- e 50 eventi in cui il K^- primario scompare in volo non dando origine ad altro che una « traccia leggera ». A 12 di tali eventi era associato un blob nel punto di sparizione del K^+ mentre gli altri 38 non presentavano tale particolarità. Si dimostra che la maggior parte degli eventi senza blob associato sono esempi di decadimenti in volo di K^-. L'analisi dettagliata di questi eventi dà i seguenti risultati: *a*) I modi di decadimento $K^-_{\mu 2}$ e $K^-_{\pi 2}$ sono da considerare come provati. Le energie cinetiche medie dei secondari osservati nel S.C.M. sono: $K_{\mu 2}$; $T_\mu^{CM} = (153 \pm 3)$ MeV, $K_{\pi 2}$; $T_\pi^{CM} = (109.5 \pm 3)$ MeV. *b*) L'abbondanza relativa dei $K_{\mu 2}$ è circa il 50%. *c*) La vita media dei K^- è $\tau_{K^-} = (1.60 \pm 0.3) \cdot 10^{-8}$ s.

(*) *Traduzione a cura della Redazione.*

Analytic Properties of the Scattering Matrix.

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Summary. — The discussion of the behaviour of the S-matrix is here carried out on the basis of the modern theory of entire analitic functions [1] for S waves and for a spinless particle scattered by a fixed spherical potential. It is shown that many of the properties of Jost's $f(k)$ functions can be derived quite rigorously and simply, from very general theorems. Among them we point out a theorem on the existence of infinite zeros of $f(k)$ when the potential decreases fast enough as to make $f(k)$ regular in the whole plane of k. The asymptotic behaviour of the distribution of these zeros is also discussed.

1. – Introductory remarks.

The functions $f(k, r)$. – In the following we shall choose units so that Schrödinger equation will be written as follows [2] (S waves):

$$\psi'' + (k^2 - V(x))\psi = 0 \,. \tag{1}$$

Following Jost we introduce two independent solutions $f(k, x)$ and $f(-k, x)$ of (1) defined by their asymptotic behaviour for large x viz,

$$\lim_{x\to\infty} \exp[ikx] f(k, x) = 1 \qquad \text{and} \qquad \lim_{x\to\infty} \exp[-ikx] f(-k, x) = 1 \,.$$

[1] R. P. Boas: *Entire functions* (New York, 1954).

[2] R. Jost: *Helv. Phys. Acta*, **20**, 256 (1947); V. Bargmann: *Rev. Mod. Phys.*, **21** (2), 488 (1949).

We define also:

$$f(x) = f(k, 0) \qquad \text{and} \qquad f(-k) = f(-k, 0)\,. \tag{2}$$

If $\int_0^\infty |V(x)|\,\mathrm{d}x$ is finite $f(k, x)$ is continuous at $k = 0$. $f(k, x)$ satisfies Green's integral equation:

$$f(k, x) = \exp[-ikx] - \frac{1}{2ik}\int_x^\infty \big(\exp[ik(x'-x)] - \exp[-ik(x'-x)]\big)\,V(x')f(k, x')\,\mathrm{d}x'\,. \tag{3}$$

(3) can be solved by iteration. It can be shown that the series of the iterations converges uniformly everywhere in the lower half complex plane of k. If $V(k)$ vanishes faster than any exponential when $x \to \infty$ this series converges uniformly for any k in any bounded region of the whole plane of k; (3) defines therefore an entire function $f(k, x)$ of k. Unless differently specified we shall assume this to be always the case. Other properties are that $\lim f(k) \to 1$ for $k \to \infty$ on the real axis and along any direction in the lower half plane. From the definition of $f(k)$, supposing that $V(x)$ is a real function, clearly follows that $f(k)^* = f(-k^*)$. From the general theory of the scattering matrix moreover the zeros of $f(k)$ (poles of $S(k)$) correspond to bound states if they lie on the lower imaginary axis, to virtual states if on the upper imaginary axis, to radiative compound states all the others. Of course if k is a zero also $-k^*$ is a zero. From the previous literature we know (Rollnik [3]) that the zeros occur in a infinite number but that only a finite number of them lies on the imaginary axis. We shall prove these properties from a more general point of view obtaining also some very accurate information on the distribution of large zeros.

2. – We shall proceed now in evaluating the asymptotic behavior of $f(k)$ in the upper half plane supposing that the potential vanishes outside a certain radius a: ($V(x) = 0$ for $x > a$). From the point of view of the modern theory of entire function there is a definite relationship between this asymptotic behaviour and the existence and distribution of zeros.

The first step is to obtain an upper limit for the iterated of eq. (3) in the upper half plane by an induction process. The starting point is the recur-

[3] H. Rollnik: *Zeits. f. Phys.*, **145**, 654 (1956).

rence formula:

$$f_{n+1}(k, x) = \frac{1}{2ik}\int_x^\infty \left(\exp\left[-ik(x'-x)\right] - \exp\left[ik(x'-x)\right]\right) V(x') f_n(k, x')\,\mathrm{d}x'. \tag{4}$$

We first observe that if $f_n(k, x)$ satisfies the following inequality:

$$|f_n(k, x)| < R_n \exp\left[2\beta\alpha - \beta x\right](a-x)^{n-1} V^{n-1}, \quad \beta = \mathrm{Im}\,(k), \quad \max|V(x)| = V. \tag{5}$$

It follows

$$|f_{n+1}(k, x)| < R_{n+1} \exp\left[2\beta a - \beta y\right](a-x)^n V^n, \qquad R_{n+1} = \frac{V R_n}{n|k|}, \tag{6}$$

R_1 is now to be evaluated. This is carried out in the appendix. The result is the following:

a) We admit that the potential can be developed near a into an asymptotic series whose principal term is $c(a-x)^\lambda$.

b) One can show then that all iterated but the first Born approximation give a negligible contribution for large k in the upper plane.

c) The principal term of the first Born approximation is then

$$f_B(k) \sim \frac{C \exp\left[-2ika\right]\Gamma(\lambda+1)}{(2ik)^{\lambda+2}}, \tag{8}$$

and this can be regarded as the principal term in the asymptotic series for $f(k)$.

From (8) we obtain as an immediate result that on any ray of the upper plane there is at most a finite number of zeros since the leading term can be made to dominate the error for sufficiently large k. This is already a generalization of Rollnik's ([3]) result.

3. – Let $F(k)$ be any analytic entire function of k and $M(h)$ the maximum modulus of $F(k)$ on the circle of center $k=0$ and radius h. The limit

$$\limsup_{h\to\infty} \frac{\log\log M(h)}{\log h} = \varrho; \quad 0 \leqslant \varrho \leqslant \infty, \tag{9}$$

is called the order of $F(k)$. The order of $f(k)$ in (8) is 1. (This is not generally true for never vanishing potentials). The type of $F(k)$ is defined as

$$\limsup_{h\to\infty} h^{-\varrho} \log M(h) = \tau. \tag{10}$$

The type of $f(k)$ is $2a$.

It is often convenient to describe shortly a function of order not larger than ϱ and of type τ if of order ϱ with the standard term « growth (ϱ, τ) ». A function of growth $(1, \tau)$, like $f(k)$, is called a function of exponential type.

There are many important theorems relating the existence and the distribution of zeros to the growth of an entire function. Before going into details it will be useful to have some definition in order to describe correctly the behaviour of the zeros. In the following we shall call k_n the zeros of $F(k)$ in order of increasing modulus. If two of them have the same modulus let the one at the right hand side have the priority. We put $h_n = |k_n|$. By $n(h)$ we denote the number of zeros of $F(k)$ in $|k| < h$. When it exists the limit $n(h)/h$ will be called the density D of the zeros. The infimum of the positive numbers α for which

$$\sum_{n=1}^{\infty} \frac{1}{h_n^{\alpha}}, \tag{11}$$

still converges is by definition the convergence exponent of the zeros of $F(k)$, it will be denoted by ϱ_1.

We shall quote here several theorems concerning entire functions:

1) ([4]) If $F(x)$ is of order ϱ, $n(h) = 0(h^{\varrho+\varepsilon})$ for every positive ε. The ε can be dropped for functions of finite type. In our case $n(h) = O(h)$.

2) ([5]) If ϱ is not an integer $\varrho_1 = \varrho$. At first this theorem seems to exclude our case. Let us take however the even function $G(k^2) = f(k)\, f(-k)$, and introduce the variable $E = k^2$; $G(E)$ is an entire function of E of growth not less than $(\frac{1}{2}, 2a)$. For this function the order $\frac{1}{2}$ must be equal to the convergence exponent. Clearly this implies the same for $f(k)$.

3) ([5]) An entire function of non-integral order has an infinite set of zeros. $G(E)$ therefore has an infinite number of zeros and so does $f(k)$. This is one of Rollnik's results.

Theorems 1), 2), 3) are quite rigorous consequences of the theory of analytic functions. Extensive use has been made in their deduction of Jensen's theorem:

4) ([6]) If $F(k)$ is regular in $|k| < H$ and $f(0) = 0$ then for $h < H$ we have

$$\int_0^h \frac{n(t)}{t}\, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \log |F(he^{i\theta})|\, \mathrm{d}\theta - \log |F(0)|. \tag{12}$$

Some less rigorous but more detailed information on the zeros of $f(k)$ can

([4]) R. P. Boas: *Entire functions* (New York, 1954), p. 15.
([5]) R. P. Boas: *Entire functions* (New York, 1954), p. 24.
([6]) R. P. Boas: *Entire functions* (New York, 1954), p. 2.

be obtained from Jensen's theorem. Actually we know that

$$(13)\qquad \begin{cases} \log\cdot f(he^{i\theta}) \sim 2ah\sin\theta - (\lambda+2)\log h\ ; & 0<\theta<\pi \\ \log\cdot f(he^{i\theta}) \sim O(1) & -\pi<\theta<0 \end{cases}$$

$$\text{for } h\to\infty\,.$$

Of course near the direction of the real axis one finds many zeros and the asymptotic behaviour (8) is there not valid. We shall neglect in (12) the corrections arising from these local irregularities. We find

$$(14)\qquad \int_0^h \frac{n(t)}{t}\,\mathrm{d}t \sim \frac{2ah}{\pi} - \frac{\lambda+2}{2}\log h\,.$$

The density of the zeros on the right (or left) hand plane is therefore a/π. This means that the distance between two successive right zeros tends to the value π/a. This result is in complete agreement with an approximate evaluation of the zeros for the square well potential. It agrees also with common sense because it states that resonances occur about any time there is an integer number of half-wave lengths in the interval $0 \ldots a$. We know already that their distance from the real axis increases less than any linear function of h. A more precise statement of this kind is given by the following theorem:

5) ([7]) If $\int_1^R (1/k^2)\log|F(k)F(-k)|\,\mathrm{d}k$ is bounded above for k on the real axis then $\sum_n^\infty \mathrm{Im}\,(1/k_n)$ converges. In our case certainly $\int_1^R$ converges because $f(k)\to 1 \quad k\to\infty$.

Now $\mathrm{Im}\,(1/k_n) = -\,(\mathrm{Im}\,(k_n)/h_n^2)$. Since $h_n^2 \sim (n\pi/2a)^2$ this implies again that $\mathrm{Im}\,(k)$ grows less than linearly in n (or h). Here as before we can resort to a less rigorous approach in order to obtain a more detailed information. To this purpose we use Carleman's theorem:

6) ([6]) If $F(k)$ is regular for $\mathrm{Im}\,(k) > 0$ and if k_n are the zeros of $F(k)$ in the upper plane and $F(0)\neq 0$ then

$$(15)\qquad \sum_{h_n<H}\left(\frac{1}{n_n}-\frac{h_n}{H^2}\right)\sin\theta_n = \frac{1}{\pi H}\int_0^\pi \log\cdot\left|\frac{F(He^{i\theta})}{F(0)}\right|\sin\theta\,\mathrm{d}\theta + \\ + \frac{1}{2\pi}\int_0^H \left(\frac{1}{x^2}-\frac{1}{H^2}\right)\log\cdot\left|\frac{F(x)F(-x)}{F(0)^2}\right|\mathrm{d}x + \frac{1}{2}\,\mathrm{Im}\,\frac{F'(0)}{F(0)}\cdot$$

$$x \text{ real}, \qquad k_n = h_n\exp[i\theta_n], \qquad \sin\theta_n > 0\,.$$

([7]) R. P. Boas: *Entire functions* (New York, 1954), p. 134.

Introducing into (5) the asymptotic expansion (8) or (13) we find that very approximately

$$\sum_{\substack{h_n<H \\ \sin\theta_n>0}} \left(\frac{1}{h_n}-\frac{h_n}{H^2}\right)\sin\theta_n \sim \frac{2a}{\pi}+\frac{1}{2}\operatorname{Im}\left(\frac{f'(0)}{f(0)}\right)+\frac{1}{2\pi}\int_0^\infty \frac{1}{x^2}\log\cdot\left|\frac{f(x)f(-x)}{f(0)^2}\right|\mathrm{d}x - \tag{16}$$

$$-\frac{(\lambda+2)\log\cdot H}{H}+O\left(\frac{1}{H}\right).$$

Letting $H\to\infty$ we find that

$$\sum_{\substack{n \\ \sin\theta_n>0}}^{\infty} \frac{\sin\theta_n}{h_n}=\frac{2a}{\pi}+\frac{1}{2}\operatorname{Im}\left(\frac{f'(0)}{f(0)}\right)+\frac{1}{2\pi}\int_0^\infty \frac{1}{x^2}\log\cdot\left|\frac{f(x)f(-x)}{f(0)^2}\right|\mathrm{d}x\,. \tag{17}$$

(16) can be written therefore as

$$\sum_{h_n>H}\frac{\sin\theta_n}{h_n}+\sum_{h_n<H}\frac{h_n\sin\theta_n}{H^2}\sim(\lambda+2)\frac{\log\cdot H}{H}\,. \tag{18}$$

From this formula the way is short to show that

$$\sin\theta_n\sim(\lambda+2)\frac{\log\cdot n}{2n}\,. \qquad \text{for} \quad n\to\infty\,. \tag{19}$$

Therefore the distance from the real axis of the zeros grows logarithmically.

5. – Expansion of $f(k)$ into an infinite product.

From the general theory we know that $f(k)$ admits a Hadamard expansion of the kind:

$$f(k)=f(0)\exp\left[-iQk\right]\cdot\prod_{n=1}^{\infty}\left(1-\frac{k}{k_n}\right)\exp\left[\frac{k}{k_n}\right]= \tag{20}$$

$$=f(0)\exp\left[-iQk\right]P(k)\,, \qquad Q \text{ real}\,.$$

This expansion converges everywhere, the point at ∞ excluded. We are now going to determine the unknown coefficients Q. From Boas' book we find that the canonical product $P(k)$ admits the following asymptotic expansion (8)

$$|P(k)|\sim\exp Ah|\sin\theta|+\operatorname{Re}\left(k\sum_n^\infty\frac{1}{k_n}\right)+hO(1)\,, \qquad A>0\,. \tag{21}$$

(8) R. P. Boas: *Entire functions* (New York 1954), pp. 148, 138.

We impose now that $f(k) \to 1$, $k \to -i\infty$. It follows:

$$\begin{cases} A = i \sum_{n=1}^{\infty} \frac{1}{k_n} + Q \,, \\ |f(k)| \sim \exp\left[Ah(\sin\theta + |\sin\theta|) + O(1)h\right] . \end{cases} \tag{22}$$

The arbitrary constant A can be determined by letting $k = i\xi$, $\xi \to \infty$ and comparing with the asymptotic expansion (8):

$$\begin{cases} a = A \\ f(k) = \exp\left[-iak\right] f(0) \prod_{n}^{\infty} \left(1 - \frac{k}{k_n}\right) . \end{cases} \tag{23}$$

6. – If we drop the restriction that the potential vanishes outside a certain finite radius a we still have to require $V(x)$ to decrease faster than any exponential function otherwise $f(k)$ will not be generally an entire function. Even with this restriction the discussion is rather complicated and it will not be carried out here. For the most common choices (Gaussian potential) there are no difficulties in following the same producedure adopted before. As a general feature the Born approximation dominates the behaviour of $f(k)$ in the upper plane. $f(k)$ is no longer of order 1 unless $V(x)$ is decreasing very fast. If $V(x) \sim \exp\left[-\tau x^{1+\varepsilon}\right]$ for large x then the order of $f(k)$ is the largest between 1 and $\left(\frac{1}{2} + (3/2\varepsilon)\right)$. In conformity with theorem 1) the zeros will not be uniformly asymptotically distributed but their relative distance will tend to decrease the faster the larger will be the growth of $f(k)$. If ε is made very small the order of $f(k)$ will become very large and its behaviour more and more pathological.

Appendix

The first Born approximation of $f(k, x)$ can be written as follows:

$$f_B(k, x) = f_0(k, x) + f_1(k, x) \,, \tag{24}$$

$$f_1(k, x) = \frac{\exp\left[ikx\right]}{2ik} \int_x^a \exp\left[-2ik\xi\right] V(\xi)\, \mathrm{d}\xi - \tag{25}$$

$$- \frac{\exp\left[-ikx\right]}{2ik} \int_x^a V(\xi)\, \mathrm{d}\xi = \frac{\exp\left[ikx\right]}{2ik} I_1 - \frac{\exp\left[-ikx\right]}{2ik} I_2 \,,$$

(9) G. Doetsch: *Teoria degli sviluppi asintotici dal punto di vista delle trasformazioni funzionali*. Pubbl. n. 420 dell'Istituto per le applicazioni del calcolo, C.N.R. (1954).

I_1 can be easily discussed in the limit $k \to \exp[i\theta]\cdot\infty$. Its asymptotic behaviour actually depends only on the behaviour of $V(x)$ near a, (12). If $V(x)$ admits the asymptotic expansion:

$$V(x) \sim c(a-x)^\lambda \qquad \text{or} \qquad V(x) = O(a-x)^\lambda\,, \tag{26}$$

from the theory of Laplace transform one can show that:

$$I_1 \sim \frac{c\exp[-2ika]\Gamma(\lambda+1)}{(2ik)^{\lambda+1}}\,. \qquad \begin{matrix} k \to e^{i\theta}\infty\,, \\ \sin\theta > 0\,. \end{matrix} \tag{27}$$

This asymptotic expansion is not uniform respect to x because for $x = a$ our integral vanishes identically. However we are for the moment interested in an upper limit only. Let $|k| = h$, $x_0 = a - \alpha/h^{1-\varepsilon}$ where α, $\varepsilon > 0$ are some constants. We have:

$$\left|\int_{x\geqslant x_0}^{a}\exp[-2ikx']V(x')\,\mathrm{d}x'\right| \sim O\left(\frac{\exp[2\beta a]}{h^{\lambda+1-\varepsilon'}}\right); \qquad \begin{matrix} \beta = \mathrm{Im}\,(k), \\ \varepsilon' = (\lambda+1)\varepsilon\,. \end{matrix} \tag{28}$$

On the other hand for $x \leqslant x_0$ our expansion is uniform. To prove it we notice that:

$$\begin{cases} I_1(x) = I_1(x_0) + \int_{x}^{x_0}\exp[-2ikx']V(x')\,\mathrm{d}x = I_1(x_0) + I_3\,, \\ |I_3| < (x_0 - x)V\exp\left[2\beta a - \dfrac{2\beta\alpha}{h^{1-\varepsilon}}\right]. \end{cases} \tag{29}$$

The last term is certainly of order small enough for k to be negligible. In both regions we have now that:

$$I_1(x) = 0\left(\frac{1}{k^{\lambda+1-\varepsilon'}}\right),$$

ε' can be small at will. We find a similar inequality for I_2. Clearly $I_2 = O(a-x)^{\lambda+1}$ for $x \to a$. For $x > x_0 = a - \alpha/h^{1-\varepsilon}$ we can write:

$$|\exp[-ikx]I_2| < \frac{M\exp[\beta x]}{h^{\lambda-1-\varepsilon'}}\,. \qquad M \text{ const}\,. \tag{30}$$

When $x < x_0$ we notice that:

$$\exp[\beta x] < \exp\left[2\beta a - \frac{2\beta\alpha}{h^{1-\varepsilon'}} - \beta x\right].$$

In the limit $k \to \exp[i\theta]\cdot\infty$; $\sin\theta > 0$; I_2 is negligible if $x < x_0$. Grouping these results together we find that for sufficently large k the following inequality holds:

$$|f_1(k,x)| < \frac{M'}{h^{\lambda+2-\varepsilon'}}\exp[2\beta a - \beta x]\,, \tag{31}$$

M is here some positive constant. Clearly similar upper bounds hold for the higher iterated:

$$|f_n(k, x)| < \frac{M'}{h^{\lambda+2-\varepsilon'}} \frac{\exp[2\beta\alpha - \beta x](a-x)^{n-1}V^{n-1}}{(n-1)!\,h^{n-1}} . \tag{32}$$

The sum of all these bounds gives an upper bound to the difference Δf between perturbed and unperturbed wave functions:

$$|\Delta f| < \frac{M''}{h^{\lambda+2-\varepsilon'}} \exp[2\beta a - \beta x] \exp\left[\frac{(a-x)V}{n}\right]. \tag{33}$$

We have now:

$$\begin{cases} f(k) = 1 - \dfrac{1}{2ik}\displaystyle\int_0^a \big(1 - \exp[-2ikx']\big) V(x')\,\mathrm{d}x' + \Delta \\ \Delta = \dfrac{1}{2ik}\displaystyle\int_0^a \big(\exp[ikx'] - \exp[-ikx']\big) V(x')\,\Delta f(x')\,\mathrm{d}x' , \end{cases} \tag{34}$$

where:

$$|\Delta| < \exp[2\beta a]\, O\left(\frac{1}{h^{\lambda+3-\varepsilon'}}\right). \tag{35}$$

And the claimed result follows.

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RIASSUNTO

Le proprietà della funzione di Jost sono derivate usando noti risultati della teoria delle funzioni intere. In particolare l'andamento asintotico dei grandi zeri di detta funzione è discusso nei dettagli.

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The Theory of Relativity, the Electromagnetic Theory and the Quantum Theory.

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Summary. — It is shown how it is possible, by means of geometry and the introduction of a principle of measurement, founded by analogy with the theory of H. Weyl, to discover a unity existing between gravitational, electromagnetic and quantum phenomena. Dirac's equation and an extension of it are derived from the principle of measurement, and an essential feature of the theory is that it incorporates a theory of the electron in which its mass appears as a geometrical quantity, entering into the equation as a consequence of the existence of a fundamental unit of length. The observed mass is given by this geometrical mass together with interaction terms which also enter naturally into the theory.

The underlying theme of this work is the union which exists between the theory of relativity, the electromagnetic theory and the quantum theory, and its purpose is to portray it by means of geometry and a theory of measurement. It will appear that the union includes the general theory of relativity but that in the applications in physics it is necessary to adopt the limitation to the special theory. This applies to the theory of the electron which is an essential feature of the union.

It has not yet appeared possible to progress in the direction of unification of these three domains of physics by the geometrical way of thinking, that is to say by appeals to the mathematical forms of geometry, as in the theory of relativity, unless an extension is made by the adoption of a continuum of

a greater number of dimensions than four. For the present purpose the number of dimensions is five. But the requirement that laws of nature must be expressed in a four-dimensional covariant form still remains an unfailing guide in the development of a physical theory. Thus, at present, whatever appeal is made to mathematical form in the search for new laws or for a new expression of known laws, the final form must satisfy this requirement. It has been found possible to ensure that in this appeal to geometry, the desired result is reached automatically. The equations obtained are either already in the correct form or are readily translated into it. Five-dimensional vectors and tensors are easily related to their four-dimensional counterparts and nothing is left which is of an artificial or unexplained character.

The advantage of this method of description is that the results acquire a simple form in the analysis and the ideas leading to them readily suggest themselves. Einstein's general theory of relativity is essentially a theory of gravitation but other phenomena, especially those of electromagnetism, suggest the search for a wider theory in which they would be united with those of gravitation.

One of these attempts at unification was made by WEYL [1]. In it he regarded electromagnetic phenomena as the revelation of a system of measurement or of gauging appropriate to the physical world, gravitational phenomena revealing an appropriate geometry. But the concept of parallel displacement, which WEYL introduced, does not appear to correspond to displacements in the physical world. Indeed, if WEYL's idea is assumed to be the basis of a physical theory, results follow which are not in agreement with experimental facts.

Another attempt was suggested by KALUZA [2], who proposed to apply the principles of Einstein's theory to a Riemannian continuum of five dimensions. The suggestion leads to a remarkable union of gravitation and electromagnetism but remains artificial since the fifth dimension is given no physical meaning. The line element of the continuum is

$$(1) \qquad d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu ,$$

where μ and ν can take the values 1 to 5. Values of the coefficient $(\gamma_{\mu\nu})$ are chosen which cause the geodesic of the continuum to represent the track of a charged particle in a gravitational and electromagnetic field.

Kaluza's theory is a geometrical one and is a theory of relativity in five dimensions analogous to that of Einstein in four. The unification it is now proposed to undertake is attempted by the adoption of Kaluza's ideas and

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[1] H. WEYL: *Raum, Zeit, Materie* (Berlin, 1921), p. 110.

[2] TH. KALUZA: *Sitzgsber. preuss. Akad. Wiss.* (1921), p. 966.

Weyl's, with differences that appear to make the resulting theory a physical one. Other suggestions for the development of a unified field theory have been made but those of WEYL and KALUZA give the background of suggestions for uniting the theories of gravitation and electromagnetism with the quantum theory.

1. – Geometrical concepts.

The relations between the coefficients of the four-dimensional line element

$$ds^2 = g_{mn}\,dx^m\,dx^n\,, \tag{2}$$

with m and n taking the values 1 to 4, and the coefficients $(\gamma_{\mu\nu})$ are given in the following table

$$\begin{cases} \gamma^{mn} = g^{mn}, \qquad \gamma^{m5} = -\alpha\varphi^m, \qquad \gamma^{55} = \dfrac{1}{\gamma_{55}} + \alpha^2\varphi_m\varphi^m\,, \\ \gamma_{mn} = g_{mn} + \gamma_{55}\alpha^2\varphi_m\varphi_n\,, \qquad \gamma_{m5} = \gamma_{55}\alpha\varphi_m\,, \end{cases} \tag{3}$$

where (φ_m) are the components of the electromagnetic potential. α and γ_{55} are independent of the co-ordinates. Certain values were given to these quantities in Kaluza's theory, α being proportional to the charge on the particle describing the geodesic and γ_{55} depending on the constant of gravitation. The values adopted here differ in both cases from those of KALUZA. The constant α is regarded as a universal constant equal to e/m_0c^2 [3], where e is the fundamental unit of charge and m_0 is identified as the rest mass of the electron.

An idea guiding the choice of values of α and γ_{55} in the present theory is that the tracks of all particles moving in external gravitational and electromagnetic fields are null geodesics [4]. Every photon in the theory of relativity describes such a path and the generalization of this is that every particle does so in the new continuum. This may well be regarded as the expression of the analogy that has been drawn between particles and waves.

To the question: Why does an uncharged particle move along a geodesic while a charged particles does not? EDDINGTON has given the reply that the charged particle deviates from a geodesic in order that the total electromagnetic fource upon it may be zero [5]. The question here is: How

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[3] H. T. FLINT: *Proc. Phys. Soc.*, **29**, 334 (1940).

[4] J. W. FISHER: *Proc. Roy. Soc.*, A **123**, 489 (1929); H. T. FLINT and E. M. WILLIAMSON: *Nuovo Cimento*, **3**, 4 (1956).

[5] A. S. EDDINGTON: *Math. Theory of Relativity*, 2nd Edition (Cambridge 1930), p. 191.

does it come about that a particle moves along a null geodesic? The answer is that it modifies the geometry in such a way as to do so. This modification is very simply expressed by means of the value of γ_{55} which depends on the number n' of unit charges e, and the multiple n of the fundamental mass m associated with the particle, the relation being [6]

$$\gamma_{55} = n'^2/n^2 , \tag{4}$$

n' is an integer but n is not necessarily integral. The reason for this choice for γ_{55} will appear later on. It is evident that γ_{55} is independent of the sign of the charge.

The line element $d\sigma$ is related to ds by the equation

$$d\sigma^2 = ds^2 + \frac{dx_5^2}{\gamma_{55}} , \tag{5}$$

where dx_5 is the fifth covariant component of the displacement, $dx_5 = \gamma_{5\mu} dx^\mu$. Writing $ds^2 = -c^2 d\tau^2$, where $d\tau$ is an element of proper-time, it appears that in order to make $d\sigma^2 = 0$, dx_5 must be identified as

$$dx_5 = \pm \frac{n'}{n} c\, d\tau , \tag{6}$$

which can be written simply as

$$dx_5 = \frac{n'}{n} c\, d\tau , \tag{7}$$

since n' can have both positive and negative integral values.

This is the physical interpretation of the fifth co-ordinate. Wherever a particle exists in the continuum $dx_5 = (n'/n)\, c d\tau$ the geometry of the space is determined by the gravitational and electromagnetic fields external to the particle. If the particle itself is regarded as a generator of a field, this field does not affect the values of the coefficients ($\gamma_{\mu\nu}$). The fields of the particles have another part to play; they are regarded as determining the system of gauging.

2. – A note on the relation between four- and five-dimensional quantities [7].

If the co-ordinates (x^m) are subject to general transformations to new co-ordinates (x'^m) independent of the fifth co-ordinate, and if this co-ordinate is transformed according to $x'^5 = x^5 + f(x^m)$, it follows that the four components

[6] H. T. Flint and E. M. Williamson: *Nuovo Cimento*, **3**, 4 (1956).

[7] H. T. Flint: *Proc. Phys. Soc.*, **29**, 417 (1940).

(A^m) of a five-vector (A^n) form a four-vector, while the covariant component A_5 is a scalar in four-dimensional analysis.

In order to distinguish between the four- and five-dimensional quantities, whenever possible, the former will be denoted by a small letter and the five-dimensional counterpart by a capital letter.
Thus

$$A^m = a^m . \tag{8}$$

Reference to the relations (5) and (6) shows that the scalar $cd\tau$ is equal to $dx_5/\sqrt{\gamma_{55}}$ and the notation.

$$A_5/\sqrt{\gamma_{55}} = a_. \tag{9}$$

is adopted for the scalar $a_.$ corresponding to A_5. By means of the relations (3) it is possible to relate other components of vectors and tensors.

Thus

$$\left\{ \begin{array}{l} A_m = a_m + \alpha\varphi_m A_5 \ = a_m + \sqrt{\gamma_{55}}\alpha\varphi_m a_. , \\ A^5 = \dfrac{A_5}{\gamma_{55}} - \alpha\varphi_m A^m = \dfrac{a_.}{\sqrt{\gamma_{55}}} - \alpha\varphi_m a^m . \end{array} \right. \tag{10}$$

Similar relations exist for tensor components.

Thus

$$A^{mn} = a^{mn} , \qquad A^m{}_5 = \sqrt{\gamma_{55}}\, a^m{}_. , \tag{11}$$

where $a^m{}_.$ is a four vector, and

$$A_{mn} = a_{mn} + \sqrt{\gamma_{55}}\,\alpha\varphi_m a_{.n} + \sqrt{\gamma_{55}}\,\alpha\varphi_n a_{m.} + \gamma_{55}\alpha^2\varphi_m\varphi_n a_{..} , \tag{12}$$

where $a_{.n}$ and $a_{m.}$ are covariant vectors and $a_{..}$ is a scalar. Also

$$A^{m5} = \frac{A^5_m}{\gamma_{55}} - \alpha\varphi_n A^{mn} = \frac{a^m{}_.}{\sqrt{\gamma_{55}}} - \alpha\varphi_n a^{mn} . \tag{13}$$

A useful relation in the case when $A^{\mu\nu}$ and $B_{\mu\nu}$ are antisymmetric tensors, $(A^{\mu\nu} = -A^{\nu\mu},\ B_{\mu\nu} = -B_{\nu\mu})$ is

$$A^{\mu\nu}B_{\mu\nu} = a^{mn}b_{mn} + 2\,a^m{}_.\,b_{m.} . \tag{14}$$

3. – Dependence on the fifth co-ordinate.

When the results of this analysis are considered, it appears that, in order to make it the basis of a physical theory, the dependence upon x^5 is a simple one. Some quantities like g_{mn} and φ_m are independent of it, and whenever

it occurs it does so in the factor $\exp[2\pi i n' x^5/l_0]$, where n' is a positive or negative integer and l_0 is a length. The integer n' is introduced into the theory in this way and is the same integer that enters into γ_{55}. Thus n' is identified as the number of fundamental charges on the particle under consideration.

The length l_0 is regarded as a fundamental length and will later be identified as the Compton wavelength $h/m_0 c$. This, it has been suggested, has the property of a minimum length. The occurrence of x^5 in this form is reminiscent of certain circuital problems in other branches of physics where there is periodicity in time and where the interest is in the fact of the periodicity and not the actual values at intervening times.

In all cases considered, x^5 occurs only as an integral multiple of l_0/n'. In the case of the electron or positron, x^5 occurs only as a multiple of l_0. No other values have physical meaning. It is suggested that this represents something fundamental in the theory and is the expression of a law of nature.

4. – A theory of measurement [8].

The unification of the quantum theory with the theories of gravitation and electromagnetism is based on the idea that this theory is essentially a theory of measurement. It is the expression of a principle of gauging analogous in its form to the theory of Weyl. In order to explain this point of view and to derive the quantum equation from it, it is necessary to express the line element of the continuum in the form of a matrix. This is done in the form:

$$\mathrm{d}\sigma = \gamma_\mu \mathrm{d}x^\mu . \tag{15}$$

The coefficients (γ_μ) are matrices and may, in general, be functions of the co-ordinates. If they satisfy the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\gamma_{\mu\nu} , \tag{16}$$

it follows that

$$\mathrm{d}\sigma^2 = \gamma_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

in agreement with the form for $\mathrm{d}\sigma^2$ already introduced.

Any vector (A^μ) will be said to have a matrix length

$$A = \gamma_\mu A^\mu \tag{17}$$

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[8] H. T. Flint: *Proc. Roy. Soc.*, A **870**, 150, 432 (1935); H. T. Flint and E. M. Williamson: *Zeits. f. Phys.*, **135**, 260 (1953).

and its length L is determined by two gauging factors θ and ψ, such that

$$L = \theta A \psi . \tag{18}$$

θ and ψ are not independent, as will be seen later on, in fact θ is the product of $\psi^\dagger$, the complex conjugate of ψ and a matrix chosen to make L a scalar quantity. The vector with components (A^μ) is said to undergo a parallel displacement when they change to $A^\mu + \mathrm{d}A^\mu$, where $\mathrm{d}A^\mu = -\Lambda^\mu_{\nu\lambda} A^\nu \mathrm{d}x^\lambda$, the coefficients $(\Lambda^\mu_{\nu\lambda})$ being the bracket expressions of the continuum of five dimensions. The result of this displacement is, in general, a change ΔL in the value of L.

If it is assumed that ΔL is linear in the displacement components $(\mathrm{d}x^\lambda)$ and the components (A^ν) for all displacements and vectors a result of the form:

$$\theta \gamma^\mu \frac{\partial \psi}{\partial x^\mu} = \theta \gamma^\mu H_\mu \psi , \tag{19}$$

follows as the equation for ψ, it being borne in mind that θ is dependent upon ψ. In Weyl's theory the place of H_μ was taken by a component of the electromagnetic potential. In the present case it is a matrix operator depending linearly upon field components. The fields are those regarded as associated with the particle itself. They may be, as in the case of the electron, electromagnetic in character, or, in the case of a nucleon, they may be nuclear fields,

The general form for H_μ is

$$H_\mu = \varepsilon_1 B_\mu + \varepsilon_2 \gamma^\nu B_{\mu\nu} + \varepsilon_3 \gamma^\nu \gamma^\varrho B_{\mu\nu\varrho} + \varepsilon_4 \gamma^\nu \gamma^\varrho \gamma^\sigma B_{\mu\nu\varrho\sigma} . \tag{20}$$

The components B_μ, $B_{\mu\nu}$, etc., are interpreted as field intensities. The quantities ε are introduced as adjustable constants. It is necessary to introduce them so that they can be chosen to cause the expression on the right of equation (19) to satisfy the requirements of covariance and further to satisfy dimensional and numerical requirements. They are thus matrices multiplied by some factor.

If there is no change in length with parallel displacement the equation to be satisfied is

$$\theta \gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0 , \tag{21}$$

which leads to the consideration of

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0 , \tag{22}$$

as the equation which determines ψ.

This is the form of Dirac's equation and suggests that ψ be regarded as identical with the ψ function of the quantum theory. The matrices (γ'') have so far been regarded as depending upon the co-ordinates, but at present no progress can be made without the limitation that they must be related to the Dirac matrices. This amounts to limiting the consideration to the case when there is no gravitational field. The electromagnetic field still remains so that the situation is that which confronts the quantum theory where the gravitational field does not influence the problem. The matrices (γ_μ) and their associated components (γ''), where $\gamma'' = \gamma''^{\nu}\gamma_\nu$, are related to their four-dimensional counterparts denoted by (β^m) and $\beta.$ by means of equations similar to (8) to (13).

Thus

$$(23) \qquad \begin{cases} \gamma^m = \beta^m, & \gamma_5 = \sqrt{\gamma_{55}}\beta., \\ \gamma^5 = \dfrac{\gamma_5}{\gamma_{55}} - \alpha\varphi_m\gamma^m = \dfrac{\beta.}{\sqrt{\gamma_{55}}} - \alpha\varphi_m\beta^m. \end{cases}$$

The function ψ will be assumed to depend upon x^5 in the usual way so that

$$\frac{\partial\psi}{\partial x^5} = \frac{2\pi i n'\psi}{l_0}.$$

After substitution of this and of the matrices given in equation (23) into (22) the following equation in four-dimensional quantities is obtained:

$$(24) \qquad \beta^m\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^m} - \frac{n'\alpha h}{l_0}\varphi_m\right)\psi + \frac{n'h}{\sqrt{\gamma_{55}}l_0}\beta.\psi = 0.$$

From equation (16) and the corresponding contravariant relation

$$(25) \qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\gamma^{\mu\nu}$$

it follows that

$$(26) \qquad \beta^m\beta^n + \beta^n\beta^m = 2g^{mn}$$

and when a gravitational field is absent the matrices (β^m) have constant components and

$$(27) \qquad \beta^m\beta^n + \beta^n\beta^m = 2\delta^{mn}.$$

In this case, by multiplication of equation (24) by $i\beta^4$ and writing

$$(28) \qquad \alpha^k = i\beta^4\beta^k, \qquad \alpha^4 = i, \qquad \beta = i\beta^4\beta.,$$

equation (24) becomes

$$\alpha^m \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^m} - \frac{n' \alpha h}{l_0} \varphi_m \right) \psi + \frac{n' h}{\sqrt{\gamma_{55}} l_0} \beta \psi = 0 , \tag{29}$$

and the α^m and β are Dirac matrices.

In classical mechanics Newton's second law of motion provides the definition of mass. In quantum mechanics the rest mass is defined by Dirac's equation. Mass is the quantity M_0 occurring in the last term of this equation in the expression $M_0 c \beta \psi$. Thus the mass is to be identified with the geometrical quantities introduced in this theory and

$$M_0 = n' h / \sqrt{\gamma_{55}}\, c l_0 . \tag{30}$$

M_0 is placed equal to $n m_0$, so that

$$n m_0 = n' h / \sqrt{\gamma_{55}}\, c l_0 . \tag{31}$$

In addition, the factor of φ_m is to be identified with $n' e / c$, so that

$$e = c \alpha h / l_0 . \tag{32}$$

From (32)

$$l_0 / e \alpha = h c / e^2 \tag{33}$$

and since hc/e^2 is a dimensionless quantity $e\alpha$ is a length r_0, such that

$$l_0 / r_0 = h c / e^2 . \tag{34}$$

From the assumption that all particles travel along the null geodesics of the continuum, it appears that α must be identified with $e / m_0 c^2$, so that it follows from equation (33) that

$$l_0 = h / m_0 c$$

and from (30) that $\sqrt{\gamma_{55}} = n'/n$.

The number n is thus to be regarded as a geometrical quantity.

5. – The significance of the lengths l_0 and r_0.

If it is accepted that, in all cases where the theory can be applied, changes in the co-ordinate x^5 less than l_0/n' have no physical significance in the study

of the motion of a particle associated with n' units of charge, it follows that

$$\mathrm{d}x^5 = \frac{\mathrm{d}x_5}{\gamma_{55}} - \alpha\varphi_n \,\mathrm{d}x^m \nless \frac{l_0}{n'},$$

or

$$\frac{c\,\mathrm{d}\tau}{\sqrt{\gamma_{55}}} - \alpha\varphi_m \,\mathrm{d}x^m \nless \frac{l_0}{n'}. \tag{35}$$

If there is no field of force, this equation gives $\mathrm{d}\tau \nless l_0/nc$. This means that an interval of proper time less than l_0/nc or h/M_0c^2, where M_0 is the mass of the particle, cannot be associated with its motion. Thus l_0 can be regarded as a minimum proper length, h/m_0c. This is a minimum length measured along the world-line of the electron.

In the case of an external static field, $\varphi_1 = \varphi_2 = \varphi_3 = 0$ and $\varphi_4 = i\varphi$, where φ is the electrostatic potential. In this case, substituting $\mathrm{d}x^4 = ic\,\mathrm{d}t$,

$$\frac{c\,\mathrm{d}\tau}{\sqrt{\gamma_{55}}} + \alpha c\varphi\,\mathrm{d}t \nless \frac{l_0}{n'}.$$

If the velocity of the particle is small, $\mathrm{d}t$ can be placed equal to the proper time $\mathrm{d}\tau$ and

$$\mathrm{d}\tau \nless \frac{h/nm_0c^2}{1 + n'\alpha\varphi/n}. \tag{36}$$

Thus the existence of φ modifies the minimum interval and tends to diminish it if $n'\varphi$ is positive, as in the case where a positive charge moves in the field of another positive charge. But if $n'\varphi$ is negative, the minimum interval increases and it would be meaningless to speak of location in time in a case in which the magnitude of $n'\alpha\varphi/n$ approaches unity. In this case another law of limitation

$$|n'\varphi/n| < 1/\alpha \tag{37}$$

is suggested.

It could be said that without this restriction it would be impossible to locate the particle on its world-line. An example is provided by an electron in the field of a positive charge, such as that of a nucleus. In the case of an electron, φ must be less than $1/\alpha$ or m_0c^2/e in order to satisfy the condition.

If a positron and electron approach one another under a Coulomb field $\varphi = e/r$, r must be greater than e^2/m_0c^2 or r_0. This gives a meaning to r_0 as a limiting length. The two limiting lengths l_0 and r_0 are related by $l_0/r_0 = hc/e^2$ in agreement with equation (34).

6. – The mass of a particle.

When a change of length occurs with a parallel displacement the quantum equation

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \gamma^\mu H_\mu \psi = 0 \,, \tag{38}$$

in accordance with equation (19), so that it can be written in the form

$$i\beta^4\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - i\beta^4\gamma^\mu H_\mu \psi = 0 \,, \tag{39}$$

in order to relate it to Dirac's equation. If the gravitational field be neglected the form taken is

$$\chi^m \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^m} - \frac{n'e}{c} \varphi_m \right) \psi + n m_0 c \beta \psi + \frac{h}{2\pi} \beta^4 \gamma^\mu H_\mu \psi = 0 \,. \tag{40}$$

Thus the mass term of Dirac's equation is replaced by the sum of two terms, and if M_0' becomes the mass defined by Dirac's equation, writing $M_0 = n m_0$,

$$M_0' c \psi^+ \beta \psi = M_0 c \psi^+ \beta \psi + \frac{h}{2\pi} \psi^+ \beta^4 \gamma^\mu H_\mu \psi \,. \tag{41}$$

The last term in this equation arises from the field due to the particle itself and, in practice, the observed mass would approximate to the mass, M_0, that has appeared from geometrical considerations, when the interaction terms are small. These terms represent an interaction of the particle's field with quantities depending on the function ψ and the matrices, which can be regarded as measuring some property of the particle.

This inclusion of the interaction terms as part of the mass of the particle is in agreement with the process of mass renormalization. In applying these ideas the case of the electron is of particular interest, but similar considerations would apply in other cases, as in that of a proton and its nuclear field.

7. – The theory of the electron.

The field associated with the electron is derived from a vector potential with four components (A_m), the fifth component A_5 being zero. It is assumed that they depend upon x^5 in the usual way. The four-dimensional counter-

parts (a_m) are the potentials usually taken to be vector potentials of the electron theory. Since $A_5 = 0$, $A_m = a_m$ (10).

The field components are ($B_{\mu\nu}$) where

$$B_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} . \tag{42}$$

For convenience k is placed equal to $2\pi m_0 e/h$ and for the electron $n' = n = 1$.

For the case $\mu = 5$, $\nu = m$, it follows from equation (42) that

$$B_{5m} = ikA_m . \tag{43}$$

In this case according to equation (20)

$$H_\mu = \varepsilon_2 \gamma^\nu B_{\mu\nu} \tag{44}$$

and the additional term in the quantum equation is

$$\frac{h}{2\pi} \beta^4 \gamma^\mu \varepsilon_2 \gamma^\nu B_{\mu\nu} \psi .$$

Thus from equation (39) it follows that

$$c\psi^+ i\beta^4 \gamma^\mu \frac{h}{2\pi i} \frac{\partial \psi}{\partial x^\mu} - \frac{hc}{2\pi} \psi^+ \beta^4 \gamma^\mu \varepsilon_2 \gamma^\nu \psi B_{\mu\nu} = 0 , \tag{45}$$

and this will be written in the form

$$c\psi^+ i\beta^4 \gamma^\mu \frac{h}{2\pi i} \frac{\partial \psi}{\partial x^\mu} - \frac{1}{4} (I^{*\mu\nu} B_{\mu\nu} + I^{\mu\nu} B^*_{\mu\nu}) = 0 . \tag{46}$$

This means that the real part of the additional term in (45) has been added to the mass term $m_0 c^2 \psi^+ \beta \psi$.

The factor of $B_{\mu\nu}$ in (45) must be a contravariant tensor of the second rank and this means that the matrix in ε_2 must be taken to be β. In this case the term contains the product $\beta^4 \gamma^4 \beta . \gamma^4$, and for the case $\mu = m$, $\nu = n$, this becomes $i\beta\alpha^m\alpha^n$, so that $\psi^+ i\beta\alpha^m\alpha^n\psi$ becomes a factor of B_{mn}. This term is a contravariant tensor of the second rank and is thus what is required. ε_2 must also be multiplied by a numerical and dimensional constant to give the term the correct magnitude and dimensions.

The expression $\mu\psi^+ i\beta\alpha^m\alpha^n\psi$, where μ is the Bohr magneton, is a well known expression for polarization density, so that it is natural to write $I^{*\mu\nu}$ in the

form:

$$I^{*\mu\nu} = g_1\mu\psi^+\beta^1\gamma^\mu\beta.\gamma^\nu\psi\ ,$$

allowing $g_1\mu$ to absorb the factor $2hc/\pi$. From this it follows that

$$I^{*mn} = I^{mn} = g_1\mu\psi^+i\beta\alpha^m\alpha^n\psi\ . \tag{47}$$

The component I_5^{*m} is important because it is a four vector. Its value is according to (47)

$$I_5^{*m} = -\,ig_1\mu\psi^+\alpha^m\psi$$

but since this is a vector quantity and I^{*mn} is a tensor, the requirements of covariance are satisfied with another constant in this case. Making use of this freedom, the value may be taken to be:

$$I_5^{*m} = -\,I_5^m = -\,ig_2\mu\psi^+\alpha^m\psi\ . \tag{48}$$

From equation (14)

$$I^{*\mu\nu}B_{\mu\nu} = i^{*mn}b_{mn} + 2i^{*m}.\,b_{m}.$$

and since

$$I^{*mn} = I^{mn}\ , \qquad i^{*mn} = i^{mn}\ .$$

Moreover

$$i^{*m}. = -\,i^{m}.\ .$$

Thus

$$\tfrac{1}{4}(I^{*\mu\nu}B_{\mu\nu} + I^{\mu\nu}B^*_{\mu\nu}) = \tfrac{1}{4}i^{mn}(b_{mn} + b^*_{mn}) + \tfrac{1}{2}i^{m}.(b^{*m}. - b^{m}.)\ ,$$

$$b^*{}_{m}. = ika_m{}^*\ , \qquad b^{m}. = -\,ika^m\ .$$

Writing

$$f_{mn} = \tfrac{1}{2}(b_{mn} + b^*{}_{mn})$$

$$f_m = \tfrac{1}{2}(a_m + a^*{}_m)$$

the expression becomes:

$$\tfrac{1}{2}g_1\mu\psi^+i\beta\alpha^m\alpha^n\psi f_{mn} - kg_2\mu\psi^+\alpha^m\psi f_m\ . \tag{49}$$

Thus the mass of the electron m_0' is given by

$$m_0'c^2\psi^+\beta\psi = m_0c^2\psi^+\beta\psi - \tfrac{1}{2}g_1\mu\psi^+i\beta\alpha^m\alpha^n\psi f_{mn} + g_2e\psi^+\alpha^m\psi f_m\ . \tag{50}$$

The electron can thus be said to possess a polarization density $\mu\psi^+i\beta\alpha^m\alpha^n\psi$, which interacts with its field with a strength measured by $-\frac{1}{2}g$, and also to possess a current density which interacts with a strength measured by g_2.

Dimensionless constants have already been introduced into the theory of the electron (9) in this way and similarly into theories of the nuclear field. In the case of the electron, values have been calculated for them.

8. – The equations of the field of the electron.

It will be assumed that the field of the electron is described by two tensors $(B_{\mu\nu})$ and $(V_{\mu\nu})$. A polarization tensor $(I_{\mu\nu})$ is introduced with

$$V_{\mu\nu} = B_{\mu\nu} - I_{\mu\nu} . \tag{51}$$

This assumption is in accord with the theory of G. Mie and is in contrast to that of H. A. Lorentz, who based his work on the assumption that the field is to be described by a single tensor $(B_{\mu\nu})$ or, as he described it, by a single electric and a single magnetic intensity. The theory of Born and Infeld also follows that of Mie.

The theory of Lorentz emphasizes the importance of the charged particle which is, in his theory, the generator of the field, so that the field is not the primary aspect of the theory. In contrast, the field is primary in the theory of Mie and the charge is a manifestation of it. In this way it may be hoped to avoid reference to the structure of the particle. In the present theory this is avoided by regarding the mass as a geometrical quantity with the addition of contributions from interaction terms, derived from the function ψ, certain matrices and constants of the theory. The tensor $(B_{\mu\nu})$ is derived from components of potential as already stated (42).

One set of field equations is thus

$$\frac{\partial B_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial B_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial B_{\lambda\mu}}{\partial x^{\nu}} = 0 . \tag{52}$$

The other set is

$$\frac{\partial V^{\mu\nu}}{\partial x^{\nu}} = 0 . \tag{53}$$

These are Mie's equations except that they are understood in the sense of five-dimensional analysis.

If $\mu = m$ (1, 2, 3, 4)

$$\frac{\partial V^{mn}}{\partial x^{n}} + ikV^{m5} = 0 , \tag{54}$$

(9) H. A. Bethe and S. Salpeter: *Handb. d. Phys.*, **35**-I, 178 (1954).

and thus becomes on substitution of V^{m5}

$$\frac{\partial V^{mn}}{\partial x^n} = -ik\left(\frac{V^5_m}{\gamma_{55}} - \alpha\varphi_n V^{mn}\right). \tag{55}$$

For the case of the electron $n' = n = 1$ so that $\gamma_{55} = 1$. In terms of four-dimensional quantities this equation becomes

$$\frac{\partial v^{mn}}{\partial x^n} = ik\alpha\varphi_n v^{mn} - ikv^m_{\cdot}\,. \tag{56}$$

The first term on the right vanishes if there is no external field, and when v^{mn} is real it can be omitted as an imaginary quantity. In general, it is to be regarded as a current density and it does, in fact, appear as a component of the energy-momentum-current tensor. It will be omitted here.

Substituting

$$v^m_{\cdot} = b^m_{\cdot} - i^m_{\cdot} = -ika^m - i^m_{\cdot}\,,$$

the equation becomes

$$\frac{\partial v^{mn}}{\partial x^n} = iki^m_{\cdot} - k^2a^m\,,$$

the form of which suggests writing $i^m_{\cdot} = -ik\chi^m$, whence

$$\frac{\partial v^{mn}}{\partial x^n} = k^2(\chi^m - a^m)\,. \tag{57}$$

The expression on the right hand side thus takes the place of a current density and is denoted by j^m.

If vectors H and D be introduced with

$$H_x = v^{23}\,, \qquad H_y = v^{31}\,, \qquad H_z = v^{12}\,,$$
$$D_x = iv^{14}\,, \qquad D_y = iv^{24}\,, \qquad D_z = iv^{34}\,,$$

together with a current density J and a charge density ϱ, with

$$J_x = j^1\,, \qquad J_y = j^2\,, \qquad J_z = j^3\,, \qquad \varrho = -ij^4\,,$$

the field equations take the form:

$$\left\{\begin{aligned} \operatorname{curl} H - \frac{1}{c}\dot{D} &= J\,, \\ \operatorname{div} D &= \varrho\,. \end{aligned}\right. \tag{58}$$

In the absence of an external field, the first set of equations (52) become

$$\left\{\begin{array}{l} \operatorname{curl} E + \frac{1}{c}\dot{B} = 0\,, \\ \operatorname{div} B = 0\,, \end{array}\right. \tag{59}$$

where

$$B_x = b_{23}\,, \qquad B_y = b_{31}\,, \qquad B_z = b_{12}\,,$$
$$E_x = ib_{14}\,, \qquad E_y = ib_{24}\,, \qquad E_z = ib_{34}\,.$$

In Mie's theory there is no specific relation given between the vectors (D, H) and (B, E) although one is assumed to exist. In the present case

$$v^{mn} = b^{mn} - i^{mn}$$

serves as a definition of v^{mn}.

If vectors I and P are introduced with

$$I_x = \quad i^{23}\,, \qquad I_y = \quad i^{31}\,, \qquad I_z = \quad i^{12}\,,$$
$$P_x = -\ ii^{14}\,, \qquad P_y = -\ ii^{24}\,, \qquad P_z = -\,ii^{34}\,,$$

the relations

$$B = H + I\,, \qquad D = E + P \tag{60}$$

follow.

9. – The Lagrangian form of the theory.

It is possible to derive the quantum equation and the field equations from the Lagrange function

$$L = -\frac{1}{4}\left(B^{*\alpha\beta}B_{\alpha\beta} - I^{*\alpha\beta}B_{\alpha\beta} - I^{\alpha\beta}B^{*}_{\alpha\beta}\right) - \frac{hc}{4\pi i}\left(\psi^{+}i\beta^{4}\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} - \frac{\partial\psi^{+}}{\partial x^{\alpha}}i\beta^{4}\gamma^{\alpha}\psi\right), \tag{61}$$

the values of the polarization components being, as before, given by

$$I^{*\alpha\beta} = g_1\mu\psi^{+}\beta^{4}\gamma^{\alpha}\beta.\gamma^{\beta}\psi\,, \qquad I^{\alpha\beta} = g_1\mu\psi^{+}\gamma^{\beta}\beta.\gamma^{\alpha}\beta^{4}\psi. \tag{62}$$

L is thus a function of ψ, ψ^{+}, $\partial\psi/\partial x^{\alpha}$, $\partial\psi^{+}/\partial x^{\alpha}$, $\partial A_{\beta}/\partial x^{\alpha}$ and $\partial A^{*}_{\beta}/\partial x^{\alpha}$. The quantum equation

$$\frac{\partial}{\partial x^{\mu}}\left\{\frac{\partial L}{\partial(\partial\psi^{+}/\partial x^{\alpha})}\right\} - \frac{\partial L}{\partial\psi^{+}} = 0\,, \tag{63}$$

is readily seen to reduce to the form (46), and the field equation

$$\frac{\partial}{\partial x_\mu}\left\{\frac{\partial L}{\partial(\partial A_\nu^+/\partial x^\mu)}\right\} = 0\,, \tag{64}$$

leads to equation (53).

The equations conjugate to (46) and (53) result from similar equations in ψ and $\partial A_\nu/\partial x^\mu$.

10. – The field tensor of energy, momentum and current.

The energy tensor $(\Theta^\mu_{\ \nu})$ is formed in the usual way from Lagrange's function:

$$\Theta^\mu_{\ \nu} = -\frac{\partial L}{\partial(\partial A_\lambda/\partial x^\mu)}\frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial L}{\partial(\partial\psi/\partial x^\mu)}\frac{\partial\psi}{\partial x^\nu} - \text{conjugate terms} + \delta^\mu_{\ \nu} L\,. \tag{65}$$

In this form the tensor is not symmetric but it can be made so by a familiar procedure. The terms added for symmetry do not affect the conservation of the tensor and do not add to the total energy. Thus, since these are the two points with which the present theory is concerned, they may be added or omitted as is convenient. The part due to the field terms will be taken in its symmetric form and the remainder will be left as it stands.

The energy tensor then becomes:

$$T^\mu_{\ \nu} = \frac{1}{2}(V^{\mu\lambda}B^*_{\nu\lambda} + V^{*\mu\lambda}B_{\nu\lambda}) - \delta^\mu_{\ \nu}\frac{1}{4}(B^{*\alpha\beta}B_{\alpha\beta} - I^{*\alpha\beta}B_{\alpha\beta} - I^{\alpha\beta}B^*_{\alpha\beta}) + \tag{66}$$

$$+\frac{hc}{4\pi i}\left(\psi^+ i\beta^4\gamma^\mu\frac{\partial\psi}{\partial x^\nu} - \frac{\partial\psi^+}{\partial x^\nu}i\beta^4\gamma^\mu\psi\right) - \delta^\mu_{\ \nu}\frac{hc}{4\pi i}\left(\psi^+ i\beta^4\gamma^\alpha\frac{\partial\psi}{\partial x^\alpha} - \frac{\partial\psi^+}{\partial x^\alpha}i\beta^4\gamma^\alpha\psi\right).$$

From the quantum equation (46) the terms in $I^{*\alpha\beta}$ and $I^{\alpha\beta}$ together with the last term in brackets vanish.

It appears from the structure of $T^\mu_{\ \nu}$ that it does not contain the co-ordinates x^5, so that the equation

$$\frac{\partial T^\nu_{\ \mu}}{\partial x^\mu} = 0\,, \tag{67}$$

which the components satisfy, becomes for the various values of ν

$$\frac{\partial T^\nu_{\ m}}{\partial x^m} = 0\,.$$

Thus the energy tensor can be regarded as (T^{μ}_{ν}) and the energy density is $-T^{1}_{4}$, the total energy being

$$W = -\int T^{4}_{4}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

From equation (66)

$$T^{4}_{4} = \frac{1}{2}(V^{*4\lambda}B^{*}_{4\lambda} + V^{*4\lambda}B_{4\lambda}) - \frac{1}{4}B^{*\alpha\beta}B_{\alpha\beta} + \frac{h}{4\pi i}\left(\psi^{+}\frac{\partial\psi}{\partial t} - \frac{\partial\psi^{+}}{\partial t}\psi\right). \tag{68}$$

If

$$u_k = \frac{h}{2\pi i}\frac{\partial}{\partial x^k} - \frac{e}{c}\varphi_k \quad \text{and} \quad u^{+}_{k} = -\frac{h}{2\pi i}\frac{\partial}{\partial x^k} - \frac{e}{c}\varphi_k, \qquad (k = 1, 2, 3),$$

$$-\frac{h}{2\pi i}\psi^{+}\frac{\partial\psi}{\partial t} = \psi^{+}(c\alpha^{k}u_k + e\varphi + m_0c^2\beta)\psi - \frac{1}{4}(I^{*\mu\nu}B_{\mu\nu} + I^{\mu\nu}B^{*}_{\mu\nu})\,,$$

k being summed over 1, 2, 3.

This is the energy density of the particle and can be regarded as made up of energy W_M consisting of kinetic and potential energy and of interaction energy W_I.

Thus

$$T^{1}_{4} = \tfrac{1}{2}(V^{4\lambda}B^{*}_{4\lambda} + V^{*4\lambda}B_{4\lambda}) - \tfrac{1}{4}B^{*\alpha\beta}B_{\alpha\beta} - W_M - W_I\,.$$

The terms can be rearranged for convenience, bearing in mind the value of W_I, so that

$$T^{1}_{4} = \tfrac{1}{2}(V^{4\lambda}B^{*}_{4\lambda} + V^{*4\lambda}B_{4\lambda}) - \tfrac{1}{4}(V^{*\alpha\beta}B_{\alpha\beta} + V^{\alpha\beta}B^{*}_{\alpha\beta}) + \tfrac{1}{4}B^{*\alpha\beta}B_{\alpha\beta} - W_M\,.$$

From the field equation it follows that

$$\int V^{*\alpha\beta}B_{\alpha\beta}\,\mathrm{d}v = -2\int\left(A_m\frac{\partial V^{*m4}}{\partial x^4} + V^{*m4}\frac{\partial A_m}{\partial x^4}\right)\mathrm{d}v\,,$$

and

$$\int\left(V^{*4\lambda}B_{4\lambda} - \frac{1}{4}V^{*\alpha\beta}B_{\alpha\beta}\,\mathrm{d}v\right. = \frac{1}{2}\int\left(A_m\frac{\partial V^{*m4}}{\partial x^4} - V^{*m4}\frac{\partial A_m}{\partial x^4}\right)\mathrm{d}v \quad (\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z).$$

Thus

$$\int\left(V^{*4\lambda}B_{4\lambda} - \frac{1}{2}V^{*\alpha\lambda}B_{\alpha\beta}\right)\mathrm{d}v = \int A_m\frac{\partial V^{*m4}}{\partial x^4}\,\mathrm{d}v\,,$$

and it is possible to write:

$$T^{4}_{4} = \frac{1}{2}\left(A_m\frac{\partial V^{*m4}}{\partial x^4} + A^{*}_{m}\frac{\partial V^{*m4}}{\partial x^4}\right) - \frac{1}{4}B^{*\alpha\beta}B_{\alpha\beta} - W_M\,. \tag{69}$$

The energy density is

$$W = W_{M} - \frac{1}{4} B^{*\alpha\beta} B_{\alpha\beta} - \frac{1}{2}\left(A^{m} \frac{\partial V^{*m4}}{\partial x^{4}} + A^{*}_{m} \frac{\partial V^{m4}}{\partial x^{4}} \right). \tag{70}$$

In the case when V^{m4} does not depend on the time and there is no magnetic field, the conditions are static and

$$W = \tfrac{1}{2} E^{2} + W_{M} .$$

The energy consists of the energy of the particle and the electrostatic energy of the field. There is no suggestion of energy of interaction and this would not be expected in any case, for if the particle gains by interaction the field must lose and the total energy is not affected by the interchange. This point may well have a bearing on the difficulties that have arisen over the interaction terms in regard to their infinite value.

The expression (70) for W must be finite, if it is to have any physical meaning. If part of it is regarded as energy of the particle and part as energy of the field, it may be that in making the division a term tending to infinity has been taken on the one hand which must be balanced by an equal term on the other. But there is no significance in this division. The interaction terms applied to the particle or to the field must be finite and, if a rule agreeing with the requirements of experiment can be found, which shows how these infinities can be avoided, it can be applied without upsetting the covariant property of the energy tensor. Such a rule would appear to be as justified as the original choice of the form of the interaction terms ([10]).

([10]) Reference should be made to the article quoted under ([9]).

RIASSUNTO (*)

Si dimostra come sia possibile, per mezzo della geometria e dell'introduzione di un principio di misura basato, per analogia, sulla teoria di H. Weyl di scoprire un'unitarietà esistente tra i fenomeni gravitazionali, elettromagnetici e quantistici. L'equazione di Dirac e una sua estensione si derivano dal principio di misura e un aspetto essenziale della teoria è di incorporare una teoria dell'elettrone nella quale la massa di questo appare come una grandezza geometrica che entra nell'equazione come conseguenza dell'esistenza di un'unità fondamentale di lunghezza. La massa osservata è data da questa grandezza geometrica assieme a termini d'interazione che a loro volta entrano naturalmente nella teoria.

(*) *Traduzione a cura della Redazione.*

Exact Determination of a Phenomenological Separable Interaction (*).

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(ricevuto il 4 Marzo 1958)

Summary. — The method of Muskhelishvili applied by Omnès to the problem of one meson production by mesons is used to solve the integral equation determining a phenomenological separable interaction from the energy dependence of an experimental phase shift in a collision problem. The interaction in momentum space is given by an expression containing only one integral. The results obtained in a previous paper concerning the existence and multiplicity of the solutions are confirmed. The method is checked by an explicit calculation in the case when the phase shift is exactly given by the shape independent approximation.

1. – Introduction.

In a previous paper (1), we have shown that the determination of a phenomenological separable interaction fitting the energy dependence of a phase shift δ obtained from the experimental study of a collision problem could be reduced to the problem of finding the solutions of the following integral equation:

$$f(k) = g(k)\left[1 + P\int_0^\infty \frac{f(p)}{p^2 - k^2}\mathrm{d}p\right], \tag{1}$$

(*) Supported in part by the United States Air Force through the European Office Air Research and Development Command.

(1) M. Gourdin and A. Martin: *Nuovo Cimento*, **6**, 757 (1957). This paper will be designated hereafter as A.

where, when we restrict ourselves to S states:

$$(2) \qquad \begin{cases} f(k) = -\dfrac{2k}{\pi} \operatorname{tg} \delta_0 , \\ g(k) = \varepsilon \dfrac{M}{2\pi^2} k^2 v^2(k) , \end{cases}$$

ε is the sign of the non-local separable interaction; $v(k)$ is the Fourier transform of the interaction.

We were able to determine solutions of this equation when $g(k)$ was any rational function of the energy k^2. Recently, however, OMNÈS [2] has proposed a very elegant method to solve an integral equation which has strong similarity with equation (1):

$$(3) \qquad \varphi(x) = \lambda(x) + \frac{1}{\pi} \int_1^\infty \frac{h^*(y)\varphi(y)\,\mathrm{d}y}{y - x - i\varepsilon} ,$$

where $h(x) = \exp[i\,\delta(x)] \sin \delta(x)$.

It can easily be seen that equation (1) can be brought into the shape (3), provided one makes the change of variable:

$$x = k^2 + 1$$

and the change of function:

$$\varphi(x) = \frac{f(x)}{g(x)} \left| 1 + i\pi \frac{g(x)}{2\sqrt{x-1}} \right| .$$

Equation (1) then becomes:

$$(4) \qquad \varphi(x) = 1 + \frac{1}{\pi} \int_1^\infty \frac{h_0^*(y)\varphi(y)\,\mathrm{d}y}{y - x - i\varepsilon} ,$$

where

$$h_0(x) = \frac{\dfrac{\pi g(x)}{2\sqrt{x-1}}}{1 - i \dfrac{\pi g(x)}{2\sqrt{x-1}}} = -\sin \delta_0 \exp[i\delta_0] .$$

[2] R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1958).

Equation (4) is therefore a special case of (3) with $\lambda(x) \equiv 1$ and $\delta_0 = -\delta$. It seems worthwhile to re-examine the solution proposed by OMNÈS because, in the special case $\lambda(x) \equiv 1$, considerable simplifications arise.

2. – Solution of equation (3) for $\lambda(x) \equiv 1$.

2·1. *Inhomogeneous equation.* – Let us first remind the method, due to MUSKHELISHVILI [3], used by OMNÈS. We define:

$$F(Z) = \frac{1}{2i\pi} \int_L \frac{h^*(y)\varphi(y)}{y - Z} \, dy ,$$

$F(Z)$ is holomorphic in the complex plane with a cut L on the real axis going from 1 to ∞ and

$$\lim_{|Z| \to \infty} F(Z) = 0 .$$

Evidently, the solution of equation (3) for $\lambda \equiv 1$ is

$$\varphi(x) = 1 + 2i F(x^+) . \tag{6}$$

OMNÈS puts $F(Z)$ in the form

$$F(Z) = \Omega(Z)\Phi(Z) , \tag{7}$$

where

$$\Omega(Z) = \exp [u(Z)] , \tag{8}$$

with

$$\begin{cases} u(Z) = \dfrac{1}{\pi} \displaystyle\int_L \frac{\delta(y)\, dy}{y - Z} , \\ u(x^+) = \varrho(x) + i\, \delta(x) , \qquad u(x^-) = \varrho(x) - i\, \delta(x) , \end{cases}$$

$$\varrho(x) = \frac{1}{\pi} P \int \frac{\delta(y)\, dy}{y - x} .$$

The function $\Omega(Z)$ is holomorphic in the complex plane except the cut L; it has no zeros and, provided $\lim_{x \to \infty} \delta(x) = 0$, $\lim_{|Z| \to \infty} \Omega(Z) = 1$.

[3] MUSKHELISVHILI: *Trud. Tbil. Mat. Inst.*, **10**, 1 (1941).

One therefore concludes that $\Phi(Z)$ must be holomorphic in the complex cut plane and go to zero in the limit $|Z| \to \infty$.

The fundamental relation deduced from eq. (3) by OMNÈS becomes

$$\Phi(x^+) - \Phi(x^-) = \sin \delta(x) \exp [-\varrho(x)] = \tag{9}$$

$$= -\frac{1}{2i} (\exp [-u(x^+)] - \exp [-u(x^-)]) .$$

A particular solution of (9) satisfying the above conditions is

$$\Phi(Z) = -\frac{1}{2i} (\exp [-u(Z)] - 1) ,$$

and therefore, using (7) and (8)

$$\varphi(x) = \exp [u(x^+)] . \tag{10}$$

We must check that (10) is a solution of the integral equation. Let us first examine the behaviour of $\exp [u(Z)]$ in the neighbourhood of $Z = 1$. We can write

$$u(Z) = N \operatorname{Log} (1 - Z) - \frac{1}{\pi} \int_L \frac{\mathrm{d}\, \delta(y)}{\mathrm{d}y} \operatorname{Log} (y - Z)\, \mathrm{d}y \tag{11}$$

if

$$\delta(0) - \delta(\infty) = -N\pi . \tag{12}$$

If we restrict ourselves to *non-negative* values of N, it is clear that $\exp [u(Z)]$ is not singular in the neighbourhood of $Z = 1$. It is then possible to write:

$$\frac{1}{2i\pi} \int_L \frac{\exp [-i\, \delta(y)] \sin \delta(y) \exp [u(y^+)]\, \mathrm{d}y}{y - x - i\varepsilon} = \frac{1}{2i} \left| \frac{1}{2i\pi} \oint_C \frac{\exp [u(Z)\, \mathrm{d}Z}{Z - x - i\varepsilon} - 1 \right| , \tag{13}$$

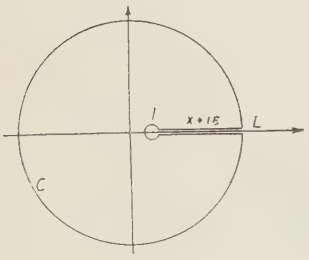

Fig. 1. – Contour C in the complex plane.

where use has been made of $\exp [u(\infty)] = 1$ and C is a contour indicated on Fig. 1.

The only contribution to the contour integral is the residue at the point $x + i\varepsilon$ and it finally turns out that

$$1 + 2iF(x^+) = \exp [u(x^+)] = \varphi(x) ,$$

which proves that (10) is a solution of the integral equation for $\lambda(x) \equiv 1$, provided

$$\delta(0) - \delta(\infty) = -N\pi \leqslant 0 .$$

It is easy to check that this solution is in agreement with Omnès' solution for $N = 0$. For $N > 1$, the comparison is not possible because the integrals appearing in Omnès' paper do not converge.

2·2. *Homogeneous equation.* – The homogeneous equation for all $\lambda(x)$ takes the form

$$\varphi_0(x) = \frac{1}{\pi} \int_L \frac{h^*(y)\varphi_0(y)}{y - x - i\varepsilon} \, dy \, . \tag{14}$$

Using relations analogous to (5) and (7) with the subscript 0 for F and Φ, Omnès obtains a function $\Phi_0(Z)$, without step in the complex plane:

$$\Phi_0(x^+) - \Phi_0(x^-) = 0 \, .$$

$\Phi(Z)$ and $\Phi_0(Z)$ must satisfy the same conditions and we obtain the general form

$$\Phi_0(Z) = \sum \frac{c_n}{(Z-1)^n} \, ,$$

and, for the solution $\varphi_0(Z)$,

$$\varphi_0(Z) = 2i \sum c_n \varphi_0^{(n)}(Z) \, ,$$

with

$$\varphi_0^{(n)}(x) = \frac{\exp\left[\varrho(x) + i\,\delta(x)\right]}{(x-1)^n} \, . \tag{15}$$

The fact that $\Phi_0(Z)$ goes to zero in the limit $|Z| \to \infty$ restricts the values of n to positive integers.

We must now verify that (15) is a true solution of (14). The right hand side (14) can be written

$$\frac{1}{\pi} \int_L \frac{\sin \delta(y) \exp\left[\varrho(y)\right] dy}{(y-1)^n (y - x - i\varepsilon)} \, . \tag{16}$$

Let us first examine the behaviour of the integrand in the neighbourhood of $Z = 1$. Using eq. (11), we define

$$\exp\left[\varrho(y)\right] = \exp\left[\varrho_{\text{reg}}(y)\right](1-y)^N \, ,$$

where $\varrho_{\text{reg}}(y)$ has no singularity for $y = 1$; $\sin \delta(y)$ is equivalent to $(y-1)^{\frac{1}{2}}$

and the convergence of the integral requires

$$N + \tfrac{1}{2} - n > -1 \qquad \text{or} \qquad n \leqslant N + 1 .$$

We can separate the possible values of n in two classes: for the first one $0 < n \leqslant N$, the integrand of (16) is a bounded function for $y = 1$; for the second one, $n = N+1$, there is a singularity at this point.

With these restrictions on n, we evaluate, as in the previous section, the integral

$$\frac{1}{2i\pi} \oint_C \frac{\exp [u(Z)]}{(Z-1)^n (Z - x - i\varepsilon)} \, \mathrm{d}Z . \tag{17}$$

The only singularity of the integrand within the region bounded by C is the pole $x + i\varepsilon$; the corresponding residue is equal to

$$\frac{\exp [u(x^+)]}{(x-1)^n} = \frac{\exp [\varrho(x) + i\,\delta(x)]}{(x-1)^n} .$$

The integral along the big circle vanishes, because $n > 0$. The two integrals along the branch cut L give the initial integral (16). If $0 < n \leqslant N$, the contribution from the little circle is zero, but when $n = N+1$, we obtain the finite value

$$\frac{\exp [u_{\text{reg}}(1)]}{1 - x} ,$$

and $\varphi_0^{(N+1)}(x)$ is not a solution of (14).

To sum up, $\varphi_0^{(n)}(x)$ is a solution of the homogeneous integral, if and only if

$$\boxed{0 < n \leqslant N}$$

with the definition of N given in equation (12).

In particular, when $N = 0$, the homogeneous equation has no solution.

3. – General solution of equation (1).

The solution of equation (3) for $\lambda(x) = 1$ reads

$$\varphi(x) = \exp [u(x^+)] \left[1 + \sum_1^N \frac{c_n}{(x-1)^n} \right] .$$

Taking into account the changes of functions and variables, we obtain the general solution of equation (1):

$$f(k^2) = -\frac{2k}{\pi} \sin \delta_0(k) \exp\left[\varrho(k^2)\right]\left[1 + \sum_1^N \frac{c_n}{k^{2n}}\right], \tag{18}$$

where $\varrho(k^2)$ is given by the integral

$$\varrho(k^2) = -\frac{2}{\pi} P\int_0^\infty \frac{\delta_0(p)}{p^2 - k^2}\, p\, \mathrm{d}p\,. \tag{19}$$

The function $f(k^2)$ must keep a constant sign (see *A*); therefore the zeros of $\sin \delta_0(k)$ and of the bracket must coincide. The bracket is an algebraic function of k^2 of degree N; it contains N unknown parameters and possesses N roots. Moreover, the relation $\delta_0(0) - \delta_0(\infty) = N\pi$ proves that $\sin \delta_0$ goes through zero at least $(N-1)$ times. There are three possibilities:

a) $\sin \delta_0$ *has* N *zeros*: they are the roots of the bracket and all of the c_n are determined. The solution is unique; this case corresponds to a system without bound state:

$$f(k^2) = -\frac{2k}{\pi} \sin \delta_0(k) \exp\left[\varrho(k^2)\right]\left[1 + \sum_1^N \frac{c_n}{k^{2n}}\right]; \tag{20}$$

b) $\sin \delta_0$ *has* $(N-1)$ *zeros*: there exists a one-parameter set of solutions corresponding to a system with one bound state; the value of the parameter is related to the binding energy (4):

$$f(k^2) = -\frac{2k}{\pi} \sin \delta_0(k) \exp\left[\varrho(k^2)\right]\left[1 + \sum_1^{N-1} \frac{c_n'}{k^{2n}}\right]\left[1 + \frac{x^2}{k^2}\right], \tag{21}$$

$x^2 = MB$, where B is the binding energy ($\hbar = 1$) and the c_n' are uniquely determined by the requirement of constant sign for f;

c) $\sin \delta_0$ *has more than* N *zeros*: there does not exist any solution with constant sign.

These conclusions are in agreement with the results of our previous work. As an example, we have calculated $f(k^2)$ by formulas (19), (20) and (21), as-

(4) We thank Mr. K. CHADAN for pointing out to us the uniqueness of the solution in the bound state case when one requires that the separable interaction must give the correct binding energy.

suming that the shape independent approximation is valid at all energies for the phase shift

$$k \operatorname{ctg} \delta_0 = -\frac{1}{a} + \frac{1}{2} r_0 k^2 .$$

$\alpha)$ $a < 0$: there is no bound state (singlet state of the nucleon-nucleon system). We write

$$\operatorname{tg} \delta = \frac{2}{r_0} \frac{k}{k^2 + \alpha^2} ;$$

it turns out that

$$\varrho(k^2) = \frac{1}{2} \operatorname{Log} \frac{k^2 + \beta_2^2}{k^2 + \beta_1^2} ,$$

where β_1^2 and $\beta_2^2 > \beta_1^2$ are given by

$$(k^2 + \alpha^2)^2 + \frac{4k^2}{r_0^2} = (k^2 + \beta_1^2)(k^2 + \beta_2^2) ,$$

which is equivalent to say that β_1 and β_2 are the roots of the equation already encountered in A:

$$\beta^2 - 2\beta/r_0 - \alpha^2 = 0 .$$

This finally gives

$$f(k^2) = -\frac{4}{\pi r_0} \frac{k^2}{k^2 + \beta_1^2} .$$

$\beta)$ $a > 0$: there is one bound state (triplet state):

$$\operatorname{tg} \delta = -\frac{2k}{r_0(k^2 - \gamma^2)} ;$$

one gets

$$\varrho(k) = \frac{1}{2} \operatorname{Log} \frac{k^4}{(k^2 + \beta_1^2)(k^2 + \beta_2^2)} ,$$

where β_1 and β_2 are the roots of

$$\beta^2 - 2\beta/r_0 + \gamma^2 = 0 ,$$

and the solution $f(k^2)$ fitting the energy of the bound state is given by

$$f(k^2) = -\frac{4}{\pi r_0} \frac{k^2(k^2 + x^2)}{(k^2 + \beta_1^2)(k^2 + \beta_2^2)} .$$

These expressions are precisely those given in Sect. **3** of A.

* * *

We thank Dr. OMNÈS for sending us a preprint of his work before publication.

RIASSUNTO (*)

Si adopera il metodo di Muskelishvili applicato da OMNÈS al problema della produzione di un mesone derivante da mesoni per risolvere l'equazione che determina un'interazione fenomenologica separabile dalla dipendenza dall'energia di uno spostamento di fase sperimentale in un problema di collisione. L'interazione nello spazio dei momenti è data da un'espressione contenente un solo integrale. I risultati riguardanti l'esistenza e la molteplicità delle soluzioni ottenuti in un precedente lavoro risultano confermati. Si verifica il metodo con un calcolo esplicito per il caso in cui lo spostamento di fase sia dato esattamente dall'approssimazione indipendente dalla forma.

(*) *Traduzione a cura della Redazione.*

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Mass Reversal and Lepton Processes (*).

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(ricevuto l'8 Marzo 1958)

Summary. — It is shown that the special mass reversals MR(1) and MR(2) can be applied to the Fermi type interactions without any internal contradiction in the theory. The selection rules obtained are consistent with the experiments. This suggests to take the universal Fermi interactions as the most probable primary interactions for all the weak interactions. From the mass reversibility of the theory under MR(1), it is concluded that the space parity is not conserved in all the weak interactions, and all the lepton processes must involve at least one neutrino whose mass must be exactly zero. From the requirement of MR(2), either a linear combination of (STP) couplings or a linear combination of (AV) couplings is shown to be allowed. The theory admits an interesting case that one can introduce two neutrinos which can play a similar role as « Twin » neutrinos, and the baryons are conserved automatically.

1. – Introduction.

We pointed out in a previous paper (1) that mass reversal is a kind of spinor transformation and is independent of charge conjugation, space reflection or time reversal. Thus special mass reversals (for example MR(1) and MR(2)) can be considered, which means that particles for which mass reversal is applied are in principle chosen arbitrarily. Hereafter we shall denote the operation of mass reversal simply by MR(n), where n is the number of part-

(*) This is the revised version of the report which appeared in *Soryushiron-Kenkyu* (mimeographed circular in Japanese), **14**, 618 (1957).

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(1) T. Ouchi, K. Senba and M. Yonezawa: *Progr. Theor. Phys.*, **15**, 431 (1956).

icles whose masses are being reversed. The weak interactions of Fermi type are taken in this paper as the most probable primary interactions responsible for the lepton processes. It is one of the main problems to examine whether these special reversals MR(1) and MR(2) can really provide the useful selection rules for the lepton processes with weak Fermi interactions.

An attempt very similar to these special mass reversals was recently proposed by NISHIJIMA [2] to derive the selection rules for lepton processes. However, we claim that the assumption of invariance under mass reversals MR(1) and MR(2) is less restrictive than Nishijima's assumptions to produce essentially the same results. The meaning of mass reversal will be made more clear and the assumptions in this paper will be justified once the structure of the elementary particles is established.

The main results obtained by applying MR(1) and MR(2) for leptons only and MR(4) for leptons and baryons are the following: Space parity is not conserved in the lepton processes. The mass of neutrinos must be exactly zero. Lepton processes must involve at least one neutrino, or pairs such as $(\mu\mu')$ or (ee'). If we furthermore try to guarantee the conservation of baryons automatically, all leptons must be fields of C or D type under mass reversal [3]. Thus we must introduce two kinds of neutrinos independently of the coupling form of Fermi interactions. These results were obtained in accordance with the baryon-meson system.

2. – Outline of the theory of mass reversal.

In order to clarify the meaning of mass reversal, it will be useful to recall Nishijima's theory [2] for leptons. He found that the following two sets of equations (I) and (II) yield the same S-matrix, but not the same Hamiltonian in quantum electrodynamics,

$$\text{(I)} \qquad \begin{cases} [\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi = 0\,, \\ \Box A_\mu = -ie\bar{\psi}\gamma_\mu\psi\,, \end{cases}$$

$$\text{(II)} \qquad \begin{cases} [\gamma_\mu(\partial_\mu - ieA_\mu) + m \exp[2i\alpha\gamma_5]\psi = 0\,, \\ \Box A_\mu = -ie\bar{\psi}\gamma_\mu\psi\,, \end{cases} \qquad (\alpha, \text{ real constant})$$

$$\text{(III)} \qquad \exp[+i\alpha\gamma_5]\gamma_\mu \exp[+i\alpha\gamma_5] = \gamma_\mu\,.$$

Next he assumed that the two sets of equations above mentioned yield the

[2] K. NISHIJIMA: *Nuovo Cimento*, **5**, 1349 (1957).
[3] T. OUCHI: *Progr. Theor. Phys.*, **17**, 743 (1957).

same S-matrix even in the weak lepton processes, and derived several interesting results which seem to be consistent with experiments.

Although the essential point of Nishijima theory is that α is arbitrary, we may fix α to $\pi/2$. Then his attempt is equivalent to the concept of mass reversal. This will be the most simple explanation of the mass reversal, but mass reversal is less restrictive than his theory in the sense that α is fixed. In fact the theory of mass reversal can be developed more generally in a similar way as space reflection.

In q-number theory of mass reversal, we have shown in the previous paper [1] that mass reversal is equivalent to a kind of spinor transformation and, therefore, can be applied even to the neutrino with zero mass. The arguments can be formulated in the interaction representation as follows:

The mass-reversibility is to assure for S-matrix the relation

$$\langle f|S|i\rangle \equiv \langle f;\varkappa|S[\varkappa]|i;\varkappa\rangle = \langle f;-\varkappa|S[-\varkappa]|i;-\varkappa\rangle = \langle f'|S'|i'\rangle\,. \tag{1}$$

where $i(i')$ and $f(f')$ represent the initial and final states respectively and S' indicate that it is constructed from a set of equations of type (II) in the same way as S is made from set (I). If there exists a time-independent unitary operator M for mass reversal satisfying $|i'\rangle = |i;-\varkappa\rangle = M|i\rangle$, eq. (1) is translated into

$$S' \equiv S[-\varkappa] \equiv MS[\varkappa]M^{-1} \tag{2}$$

or in terms of interaction hamiltonian H,

$$H' \equiv H[-\varkappa] = \varrho_H(MH[+\varkappa]M^{-1})\,. \tag{3}$$

And in general for any physical quantities Q's we have

$$Q' \equiv Q[-\varkappa] = \varrho\,(MQ(\varkappa)M^{-1})\,, \tag{4}$$

where ϱ is a sign function for Q under mass reversal. In the usual case ϱ_H must be taken as $+1$ as H is energy. $H(\varkappa)$ contains in general a certain number of explicit masses $\varkappa_i$ and fields $\psi_i(\varkappa)$. Then $H(-\varkappa)$ has the same form as $H(\varkappa)$, but with explicit masses $-\varkappa_i$ and fields $\psi_i(-\varkappa)$ instead of $\psi_i(\varkappa)$. Let us take as the most general interactions of Fermi type

$$H = \sum_{\lambda} f_\lambda(\bar\psi^a O_\alpha \psi^b)(\bar\psi^c O'_\alpha \psi^d) \tag{5}$$

and take into account the special mass reversals MR(1) and MR(2) together with the usual MR(4). This possibility was pointed out in the previous

paper (1) from the physical meaning of mass reversal. As the double MR(2) is considered physically to be equivalent to MR(4) and double MR(1) is essentially equivalent to MR(2), we assume the value of ϱ_H for MR(2) and MR(1) to be (± 1) and $(\mp 1, \mp i)$ respectively. Corresponding to this, eq. (1) should be modified slightly. The meaning of these assumptions will be clarified in the next section with an example (see eq. (19)).

As was shown in the previous paper (1), the operator M exists and the spinor fields ψ_i and bose fields φ_i transform as follows:

$$(6) \qquad \begin{cases} \psi_i M(x, \varkappa_i) M^{-1} = \omega_i \gamma_5 \psi_i(x, -\varkappa_i), \\ M\varphi_i(x, \varkappa_i) M^{-1} = \omega_i' \varphi_i(x, -\varkappa_i), \end{cases}$$

where ω_i is the relative phase of ψ_i under mass reversal and takes the value ± 1, $\pm i$, and ω_i' is the mass parity of bosons, taking the value ± 1. Now we can derive the selection rules from the requirements of mass reversibility of the hamiltonian. Assuming that the f_λ do not involve any explicit mass, we get from eqs. (3), (5) and (6),

$$(7a) \qquad \omega_a^* \omega_b \omega_c^* \omega_d = 1, \qquad \text{from MR(4),}$$

$$(7b) \qquad \pm \omega_c^* \omega_d = \alpha, \quad \text{with} \quad \alpha = \pm 1, \qquad \text{from MR(2),}$$

where we take the $+$ sign for O_α' with V or A coupling and the $-$ sign for O_α' with S, T or P coupling.,

$$(7c) \qquad O_\alpha' \gamma_5 \omega_\alpha = \beta O_\alpha' \quad \text{with} \quad \beta = \pm 1, \pm i \qquad \text{from MR(1).}$$

As ω_α is limited to the value ± 1 and $\pm i$, eq. $(7c)$ means

$$(7c') \qquad O_\alpha' = O_\alpha(1 \pm \gamma_5), \quad \text{or} \quad O_\alpha' = O_\alpha(1 \mp i\gamma_5).$$

$(7b)$ means that there occurs either a linear combination of S, T and P couplings or a linear combination of V and A couplings, which are the same results obtained by TIOMNO (4). $(7c')$ means that space parity is not conserved in the weak Fermi interactions.

3. – Application of various mass reversals to leptons.

Now we are ready to apply all kinds of mass reversal to Fermi type interactions due to lepton processes. As the form invariance of the theory under MR(1), MR(2) and MR(4) is simultaneously required, eq. (1) should be replaced

(4) J. TIOMNO: *Nuovo Cimento*, **1**, 226 (1955).

correspondingly by

$$|\langle f;\varkappa|S[\varkappa]|i;\varkappa\rangle|^2=|\langle f;-\varkappa|S[-\varkappa|i;-\varkappa\rangle|^2\,, \tag{9}$$

where $|.....|^2$ implies that one takes care of the average over the initial spin states and the summation over the final spin states. The above change is necessary because α and β for MR(2) and MR(1) are not always equal to $+1$.

It is easy to derive from MR(1) a theorem which states that in lepton processes the space parity is not conserved and no pair of (μ, e) appears, and the mass of the neutrino must be exactly zero. This theorem can therefore forbid such processes as $\mu\to\mathrm{e}+\mathrm{e}+\mathrm{e}$ or $\mu\to\mathrm{e}+\gamma$, which were excluded by KONOPINSKI and MAHMOUD [5] by assuming the conservation of leptons. Therefore the lepton processes must involve at least one neutrino, or else produce a pair such as (μ, μ') or $(\mathrm{e}, \mathrm{e}')$, where $\mu'(\mathrm{e}')$ is the anti-particle of $\mu(\mathrm{e})$ or a particle with same mass as $\mu(\mathrm{e})$. Parity non-conservation due to MR(1) does not necessarily mean that the neutrino should be described by a two-component Dirac field. Therefore a slight difference can be expected in certain cases between the Lee-Yang theory [6] and ours. This point was examined recently by FUJII and IWATA [7].

Here the proof will be illustrated with the example of $\pi\to\mu+\nu$ decay. From our standpoint this decay is considered to take place through the baryon loops. Denoting the wave functions for pion, muon and neutrino by $\varphi(\pi)$, $u(\mu)$, and $u(\nu)$, we can show easily

$$\langle f|S|i\rangle\propto[u^+(\mu)O'u(\nu)]\varphi(\pi)\,, \tag{10}$$

where O' is a linear combination of γ-matrices and is of the form $O(1\pm\gamma_5)$ or $O(1\pm i\gamma_5)$. As the application of MR(1) to the particles which form the loop is essentially equivalent to that of MR(2), the contribution of the loop is not written explicitly in eq. (10). Then after the summation and the average over spin directions, we get

$$|\langle f|S|i\rangle|^2\propto \mathrm{Sp}\{(i\gamma p+\varkappa_\mu)O'(i\gamma p'+\varkappa_\nu)O'^+\}, \tag{11}$$

which, from the requirement of MR(1), namely eq. (9), must be equal to

$$\begin{aligned}\mathrm{Sp}\{(i\gamma p-\varkappa_\mu)O'(i\gamma p'+\varkappa_\nu)O'^+\}&=\\ =\mathrm{Sp}\{-\gamma_5(i\gamma p+\varkappa_\mu)\gamma_5 O'(i\gamma p'+\varkappa_\nu)O'^+\}&=\\ =\mathrm{Sp}\{(i\gamma p+\varkappa_\mu)O'(i\gamma p-\varkappa_\nu)O'^+\}&.\end{aligned} \tag{12}$$

(5) E. J. KONOPINSKI and H. M. MAHMOUD: *Phys. Rev.*, **92**, 1045 (1953).
(6) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957).
(7) K. FUJII and K. IWATA: *Progr. Theor. Phys.*, **18**, 666 (1957).

Comparing (11) with (12), we can conclude that the mass of the neutrino $\varkappa_\nu = 0$, because O' is taken as a linear combination of γ-matrices. In a similar way we can forbid the process $\mu \to e+e+e$ or $\mu \to e+\gamma$, etc., from MR(1). But the pair production of (μ, μ') (e, e') cannot be forbidden because MR(1) for these transitions is just MR(2) and the above argument is not valid.

Next we shall derive Nishijima's statement from the mass reversibility for MR(2): « All the weak Fermi interactions have always a linear combination of S, T and P couplings ». To assure the consistency of the theory of mass reversal, the mass parity assignment for baryons and mesons is necessary. We have formerly proposed [3] the form of the strong interactions and the weak interactions for the hyperon and meson system. As the most probable primary interactions of the Yukawa type for this system, we suggested

$$(13) \qquad \begin{cases} G\bar{\psi}_a \gamma_5' \psi_b \varphi_c & \text{for the strong interactions,} \\ \text{and } g\bar{\psi}_a \gamma_5' \gamma_\mu \psi_b' \partial_\mu \varphi_c & \text{for the weak interactions,} \end{cases}$$

where G and g are assumed to be natural constants with the magnitude of order $1 \div 10$ and $10^{-13} \div 10^{-14}$ in units of $\hbar = c = \mu$ ($= pion\ mass$) similar to e for the electromagnetic interaction and γ_5' is taken to be either γ_5 or 1 if the space parity is conserved in the weak interaction, but all γ_5' can be put generally equal to $1 \pm \gamma_5$ if the space parity is not conserved in the weak interactions. Assigning the mass parity as follows:

$$(14) \qquad \Xi,\ N \in A, \qquad \Sigma,\ \Lambda \in B, \qquad \pi \in -1, \qquad K \in +1$$

where $\in$ mean that Ξ, N belong to the fields of type A under mass reversal etc., we can derive good selection rules which are consistent with experiment. If we connect the mass parity with the attribute a [8] so that $\omega = (-1)^a$ for fermions and $\omega = -(-1)^a$ for bosons, our selection rule reads

$$(15) \qquad (-1)^{\sum_i a_i} = (-1)^{\sum_f a_f} (-1)^{n.w.}.$$

where i and f indicate the initial and final states, the summation should be done over all the particles present in the initial and final states and $n.w.$ means the number of vertices with weak coupling g in the Feynman diagram. From (15), we can derive the even-odd rule,

$$(16) \qquad \begin{cases} \Delta a = 0 \pmod 2, & \text{for the strong interactions,} \\ \Delta a = 1 \pmod 2, & \text{for the weak interactions.} \end{cases}$$

[8] R. G. Sachs: *Phys. Rev.*, **99**, 1573 (1955). It is an important point for the derivation of Sachs' rule that g does not depend on the mass of boson.

Sachs' rule can be derived from eq. (15) less rigorously but more naturally than the even-odd rule,

$$\Delta a = n \equiv n.\ w.\ . \tag{17}$$

Therefore we may regard eq. (17) as restriction on the form of interaction Hamiltonian under consideration. Although SACHS assumed that the transition having the larger n becomes much weaker, the strength of the transitions having $\Delta a = n$ can be estimated in our theory to be of order g^n, therefore very weak unless $n = 0$ or 1. This is the reason why there exist the strong and weak interactions observed in the hyperon-meson system.

From the analysis of β-decay, and that of K^+_{e3}-decay by the Hiroshima group ([9]), T-coupling seems to be necessary for the primary Fermi interactions. Taking this fact into account, we take the — sign in eq. (7*b*) and we have consistently with (7*a*) and (14)

$$-\omega_e^*\omega_\nu = \alpha = -1\,, \qquad -\omega_e^*\omega_{\nu'} = \alpha' = 1\,, \tag{18}$$

where ν' denotes the neutrino produced in K_{e3}-decays and is distinguished from ν.

We can have various solutions for ω_i from eqs. (18) and (7*a*).

Case A. If we assume the existence of two kinds of neutrinos, they are both particles and belong to different types under mass reversal. And as it can be easily shown, the sign of α is intimately related to the kind of neutrinos which appear in all the Fermi interactions. If we accept Nishijima's statement, we can take, most naturally as,

$$\begin{cases} \alpha = -1\,, & \text{when } \sum_i S_i = \text{even}\,; \quad \text{neutrino } \nu \text{ is produced,} \\ \alpha = +1\,, & \text{when } \sum_i S_i = \text{odd}\,; \quad \text{neutrino } \nu' \text{ is produced,} \end{cases} \tag{19}$$

where the summation runs over the strange particles contained in the interaction and S indicates the strangeness of the particle. This rule cannot be applied immediately for μ-e decays, because no strange particles are included in this decay. If we can determine the coupling type of μ-e decay and the kind of neutrinos contained in it, we can decide whether the conservation of leptons is valid or not. If we further assume all leptons to be *C* or *D* type field under mass reversal, the conservation of baryons is guaranteed auto-

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([9]) S. FURUICHI, S. SAWADA and M. YONEZAWA: *Nuovo Cimento* (to be published).

matically in the same way as treated by YANG and TIOMNO ([1]). If the parity non-conservation can be ascribed to the property of neutrinos, and the relation (19) is abandoned, these neutrinos seem to play a similar role as the « twin » neutrinos ([11]). This interesting point will be examined elsewhere. The introduction of a redundant neutrino might be a defect of the present theory, but it will not be taken seriously because it originates from the formal ambiguity of mass reversal due to the vanishing rest mass of the neutrino.

Case B. If there is only one kind of neutrino, neutrino ν' must be the anti-particle of ν having the transformation property A or B under mass reversal. Therefore the conservation of leptons does not hold. But, as long as we support Nishijima's rule, α is shown to be related with the strangeness S as shown by (18).

It should be pointed out that the above solutions case A and case B are obtained on the assumption that the coupling constants f_λ are independent of each other, and do not involve the explicit masses. Without this assumption, we can have many solutions. In this case it is possible to do similar tricks as SALAM ([12]) and explain the non-concurrence of $\pi \to e+\nu$ decay. But the situation seems to be very complicated and its systematic examination will be published elsewhere.

4. – Concluding remarks.

The advantage of the theory of mass reversal over the Nishijima theory, is that the various points are explained in a unified manner in the theory of mass reversal. For example, we need not assume the Lee-Yang neutrino to derive Sachs' assumption that the lepton processes must involve at least one neutrino. And we have more freedom to determine the coupling forms of Fermi interactions than in Nishijima's theory. The parity non-conservation does not always mean in our theory that the observed neutrino should be described by a two component Dirac field. From the standpoint of MR(1), FUJII and IWATA have examined the polarization problem of the β-decay and μ-e decay and they found that there exists a slight difference in the case of β-decay between the two-component neutrino theory and ours. And it would be possible to detect this difference when more precise experiments are done, although so far both theories explain the experiments well. Furthermore, we have shown that the conservation of hyperons seems to be related closely with the existence of the « twin » neutrinos.

([10]) C. N. YANG and J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950).
([11]) J. G. MAYER and V. L. TELEGDI: *Phys. Rev.*, **107**, 1445 (1957).
([12]) A. SALAM: *Nuovo Cimento*, **5**, 299 (1955).

As is easily understood, the special mass reversal MR(1) can be applied to Sakata's ([13]) composite model for the baryons and mesons, and we can find that the space parity is conserved for the strong interactions, but is violated for the weak interactions. In other words, all the Fermi interactions (including leptons) having only an even number of particles with same masses in themselves are strong and the others are week. One remarkable fact is that the application of MR(1) to Yukawa-type interactions (13) which are essentially the same as that of Sakata theory gives different results. This facts shows therefore that there is a great difference between Fermi interactions and the Yukawa ones, and that the mass reversal MR(1) is related deeply with the Fermi interactions.

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The authors would like to express their sincere thanks to Prof. K. SAKUMA and Dr. S. OGAWA for their interests in this work.

([13]) S. SAKATA: *Progr. Theor. Phys.*, **16**, 686 (1956).

RIASSUNTO (*)

Si dimostra che le inversioni di massa speciali MR(1) e MR(2) si possono applicare alle interazioni di tipo Fermi senza alcuna contraddizione interna nella teoria. Le regole di selezione ottenute sono compatibili con l'esperienza. Ciò suggerisce di ritenere le interazioni universali di Fermi le più probabili interazioni primarie fra tutte le interazioni deboli. Dalla reversibilità di massa della teoria rispetto ad MR(1) si conclude che la parità spaziale non è conservata in tutte le interazioni deboli e tutti i processi coinvolgenti leptoni debbono coinvolgere almeno un neutrino la cui massa deve essere esattamente nulla. Dal requisito di MR(2) si dimostra che è permessa o una combinazione lineare di accoppiamenti (STP) o una combinazione lineare di accoppiamenti (AV). La teoria ammette un caso interessante nel quale si possono introdurre due neutrini che possono avere una funzione simile come neutrini « gemelli » e i barioni sono conservati automaticamente.

(*) *Traduzione a cura della Redazione.*

On the Clearing of Emulsion Plates.

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(ricevuto il 26 Marzo 1958)

Summary. — The effect of the addition of Ammonium Chloride, Ammonium Hydroxide, and Ammonium Sulphocyanide to the Hypo bath of different concentrations, on the clearing time of thin and thick emulsions (up to 1000 μm) has been observed. In the case of thin emulsions no rise for higher concentrations has been found. A detailed study of the three baths, for different concentrations of Ammonium salts has been made, and the results indicate an optimum value for a particular temperature and thickness. The effects of the Ammonium baths on the developed grain size has also been thoroughly studied. The results have been discussed on the basis of the prevalent theory. The use of Ammonium Sulphocyanide is favoured in comparision to the other two salts.

1. – Introduction.

The clearing time of emulsion plates increases considerably as the thickness of the plates is increased. These long times always increase the probability of some distortions being created in the emulsion plates during the processing (Burge *et al.* [1]). Attempts have been made to reduce the clearing time of these emulsions and it was found that 35% Sodium Thiosulphate (Y. Prakash, 1955 [2]) gives the minimum time for the clearing of emulsions up to 400 μm. Some authors have recommended the use of Ammonium Chloride in fixing bath which reduces the time to some extent (Dainton *et al.*, 1951 [3]);

(1) E. J. Burge, J. H. Davies, I. J. van Heerden and D. J. Prowse: *Nuovo Cimento*, **5**, 1005 (1957).

(2) Y. Prakash: *Ind. Journ. Phys.*, **29**, 569 (1955).

(3) A. D. Dainton, A. R. Gattiker and W. O. Lock: *Phil. Mag.*, **42**, 396 (1951).

STILLER *et al.*, 1954 ([4]); WILSON, VANESLOW, 1948 ([5]); etc.), but they have reported that the addition of Ammonium Chloride affects the developed grains of silver at the surface of the plates.

It has been pointed out (MEES, p. 523) that the decrease in the clearing time by the addition of the Ammonium Chloride to Hypo is due to the fact that Ammonium Thiosulphate (Eqn. (1)) is a better complex making salt as compared to Sodium Thiosulphate (Hypo):

$$2NH_4Cl + Na_2S_2O_3 \rightleftharpoons (NH_4)_2S_2O_3 + 2NaCl\,. \tag{1}$$

But the presence of the Chlorine ion in such additions helps in the oxidation of silver grains and this may be responsible for the eating up of the developed grains near the surface as observed by the above authors.

In view of these observations it was thought useful to make a systematic study of Ammonium Thiosulphate bath and various salts, viz. Ammonium Chloride, Ammonium Hydroxide and Ammonium Sulphocyanide. Ammonium Hydroxide and Ammonium Sulphocyanide were tried with a view to avoid the Cl ion concentration which was thought to be responsible for the eating up of the surface grains and slight rise in the clearing time at higher concentrations. Sodium Sulphocyanide (Eqn. (2)) resulting by the mixture of Ammonium Sulphocyanide and Sodium Thiosulphate, itself acts as a complex making reagent and was considered to be more useful than the other two,

$$Na_2S_2O_3 + 2NH_4SCN \rightleftharpoons (NH_4)_2S_2O_3 + 2NaSCN\,. \tag{2}$$

2. – Experimental details and results.

2·1. *Thin plates.* – PIPER ([6]) (MEES, p. 523) has shown the effects of the addition of Ammonium Chloride to Hypo bath of different concentration on the clearing time of thin emulsions (ordinary photographic plates and films etc.). Fig. 1 reproduces the curves given by him. All the curves indicate the same nature and an optimum value of Ammonium Chloride to be added to a hypo bath of a particular concentration. The clearing time increases on both the sides of this optimum value. It has been pointed out that the first part of any curve (the fall in the clearing time) may be explained due to the rapid rate of the complex formation by Ammonium Thiosulphate as compared to Sodium Thiosulphate, which decreases the clearing time of the emulsion plates.

([4]) B. STILLER, M. M. SHAPIRO and F. W. O'DELL: *Rev. Sci. Instr.*, **25**, 340 (1954).
([5]) M. Y. WILSON and W. VANSELOW: *Phys. Rev.*, **75**, 1144 (1949).
([6]) PIPER and BOOK: *Fundamentals of Photographic Theory*, p. 523.

The latter part of the curves (increase in the clearing time) may be explained due to the combined effects of two factors: (i) the increase of « Cl » ion concentration affects the diffusion rate and may supress the reaction; (ii) the higher salts of Ammonium Thiosulphate and Silver Halide complexes may be unstable as compared to the higher complexes of Sodium Thiosulphate and Silver Halide.

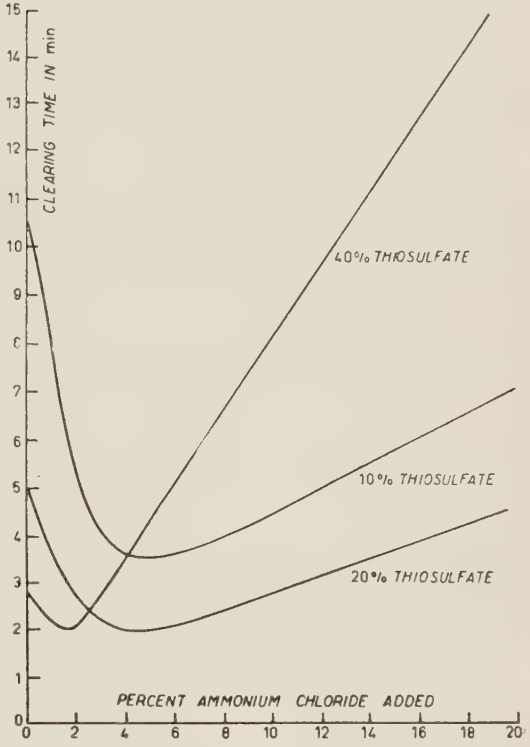

Fig. 1. – Reproduction of the curves of Piper showing the variation of clearing time vs. the concentration of Ammonium Chloride for 40, 20, and 10% Hypo baths. (K. Mees: *Photographic Theory*, p. 523).

We have carried out similar experiments for the thin plates for all the above chemicals, viz.: Ammonium Chloride, Ammonium Hydroxide, and Ammonium Sulphocyanide. The plates of the same size were put directly in the fixing bath maintained at the room temperature and the clearing times were noted. We define the clearing time as the time at which the last grain of Silver Halide becomes invisible. No agitation was done during the process. The same quantity of Hypo solution was used in one experiment so that the layer of Hypo above the plates remained constant in all the cases. The plates from the same batch were taken in each experiment and an average has been considered. The percentage of Ammonium Chloride and other salts was calculated on the basis of the volume of the Hypo solution taken. The results have been shown in Fig. 2 and Fig. 3 (Ammonium Chloride added to 10, 20 and 35% of hypo). Fig. 2 shows the curves for all the three salts added to 35% Hypo. In most of the cases only one concentration of Hypo (35%) has been considered because the knowledge of the behaviour at this percentage was more important for the processing of thick emulsions.

The curves for Ammonium Chloride (Fig. 3) are quite different than those given by Piper (Fig. 1). We are getting a decrease in time as the concentration of Ammonium Chloride increases, but are not getting a marked increase

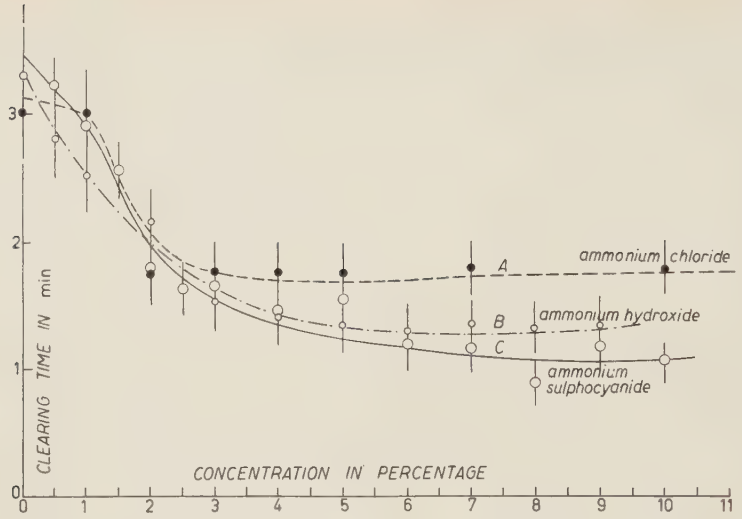

Fig. 2. – Clearing time of thin plates in minutes vs. the concentration of Ammonium salts added to a Hypo bath of 35% at room temperature (~ 15 °C). Curve *A* is for Ammonium Chloride, *B* for Ammonium Hydroxide, and *C* for Ammonium Sulphocyanide.

at higher concentrations as indicated by the curves of Fig. 1. All our curves become flat after reaching a minimum value, and the rise in the clearing time,

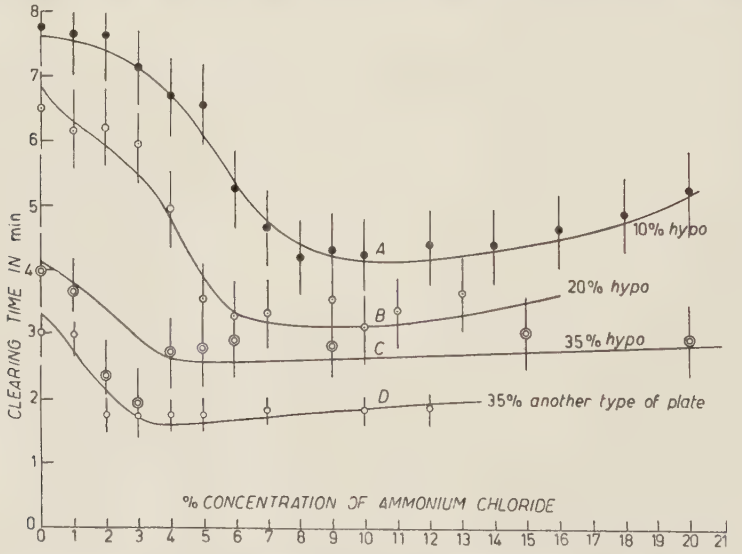

Fig. 3. – Clearing time of thin plates (Ilford, Hypersensitive) in minutes vs. the concentration of Ammonium Chloride added to different concentration of Hypo bath. Curve *A* is for 10% Hypo, *B* for 20%, and *C* for 35%. Curve *D* shows the clearing time for another type (Ortho plates) in a 35% Hypo bath. The temperature was kept at about 15 °C.

if at all, is very nominal. The curves *A*, *B*, *C* in Fig. 3 for the different concentrations of Hypo are showing the same nature and are almost parallel to each other within statistical errors. The nature of the curves remains unchanged even if different plates are used as indicated by the curves 3*C* and 3*D* where two different plates of the same size were tested in 35% Hypo solution. In all the cases the fall is about 50% as compared to the time taken in pure Hypo. The position of the minimum changes with the concentration of the Hypo solution and is about 9-10%, 6-7% and 4-5% for 10%, 20% and 35% Hypo solution respectively. The minimum does not change with the nature of the plate and also remains unchanged for Ammonium Hydroxide and Ammonium Sulphocyanide (Fig. 2). The curve for Ammonium Hydroxide is much more flat than that of Ammonium Chloride, while there is a downward trend in the case of Ammonium Sulphocyanide even up to 10%. The percentage fall in clearing time as compared to the time of 35% Hypo bath, is about 48% in the case of Ammonium Chloride, 53% in the case of Ammonium Hydroxide, and 60% in the case of Ammonium Sulphocyanide.

2·2. *Thick plates.* – Experiments similar to thin plates have been carried out for thick emulsion plates also. All the three salts viz: Ammonium Chloride, Ammonium Hydroxide, and Ammonium Sulphocyanide added to a Hypo bath of 35% were tried. The clearing times were noted in a similar manner as described above. The plates were placed in the clearing bath directly prior to development or presoaking. The temperatures of the Hypo baths were maintained around 8 °C by a thermostatically controlled cooling cabinet which is capable of controlling the temperature up to ± 1 °C. No agitation was done during the clearing process. For a single experiment plates (Ilford, G5), from the same batch were used and the quantity of Hypo in each case was kept constant.

The results obtained for 50, 100, 200, 400 and 600 μm plates have been given in Fig. 4, 5 and 6, while Fig. 7 shows the percentage reduction obtained in the case of Ammonium Chloride, Ammonium Hydroxide and Ammonium Sulphocyanide respectively. The curves of Fig. 4-6 are similar for plates of different thicknesses and exhibit the same trend for three different baths. The curves indicate a fall in the clearing time as the concentration of Ammonium salts is increased and later at higher concentrations either become constant or show a small increase from the minimum. This increase is found only in the case of Ammonium Chloride and sufficiently marked in the case of 200 μm plates. The rise in time in the case of Ammonium Chloride is decreased as the temperature of the bath is increased. Also under similar conditions of temperature etc., similar types of plates (Ilford G5, 200 μm) Ammonium Hydroxide shows the minimum fall in the clearing time—being about 5%—while Ammonium Chloride and Ammonium Sulphocyanide indi-

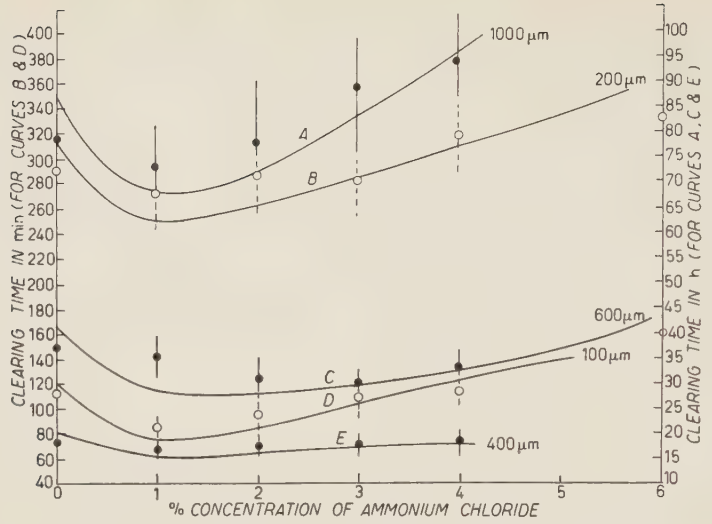

Fig. 4. – The clearing time of thick plates (Ilford, G-5) vs. the concentration of Ammonium Chloride. Scale of minutes stands for curves B and D, while that of hours for A, C, and E. Curves A, B, C, D and E show the clearing time of 1000, 200, 600, 100 and 400 μm plates respectively for a temperature of about 8 °C.

cate a fall of about 10-12% and 15% respectively. The same is true for the plates of higher thicknesses. These results have been summarized in Fig. 7.

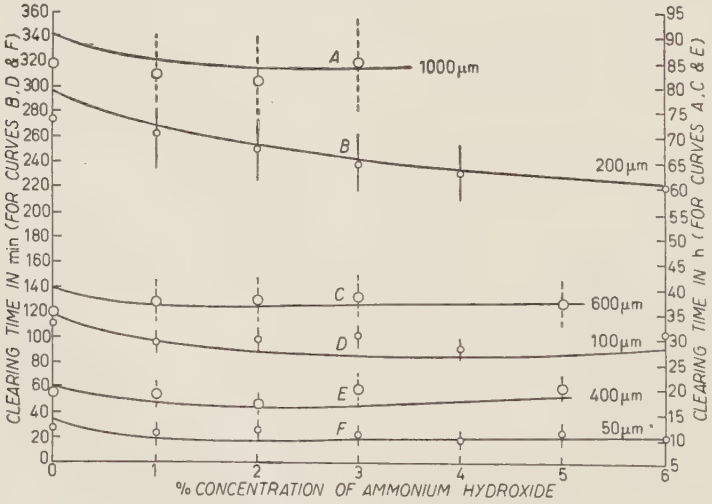

Fig. 5. – Clearing time of G-5 emulsion plates vs. the concentration of Ammonium Hydroxide. Curves A, B, C, D, E and F represent the time for the thicknesses noted in each case. Scale of minutes stands for curves B, D and F, while that of hours for A, C and E. The temperature was about 8 °C.

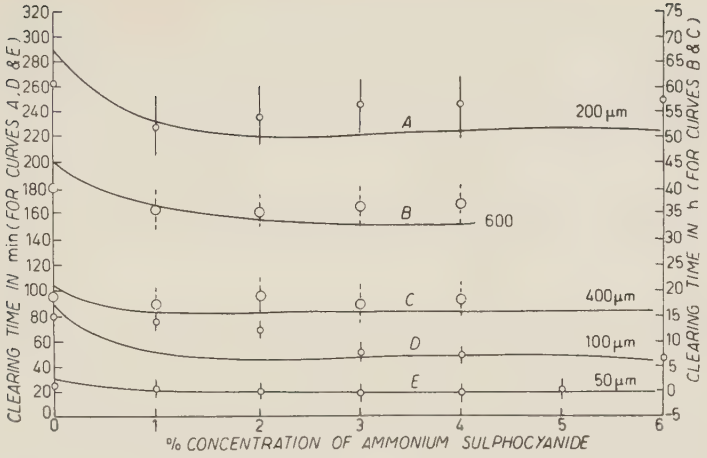

Fig. 6. – Clearing time of G-5 emulsion plates vs. the concentration of Ammonium Sulphocyanide. Curves *A*, *B*, *C*, *D* and *E* represent the time for 200, 600, 400, 100 and 50 μm respectively. Scale of minutes stands for curves *A*, *D* and *E*, while that of hours for *B* and *C*. The temperature was about 8 °C.

In all the three curves of this figure the value of the percentage fall suddenly changes as we change from thin to thick emulsions but the value of this fall becomes fairly constant after 200 μm. Critical examination reveals that the

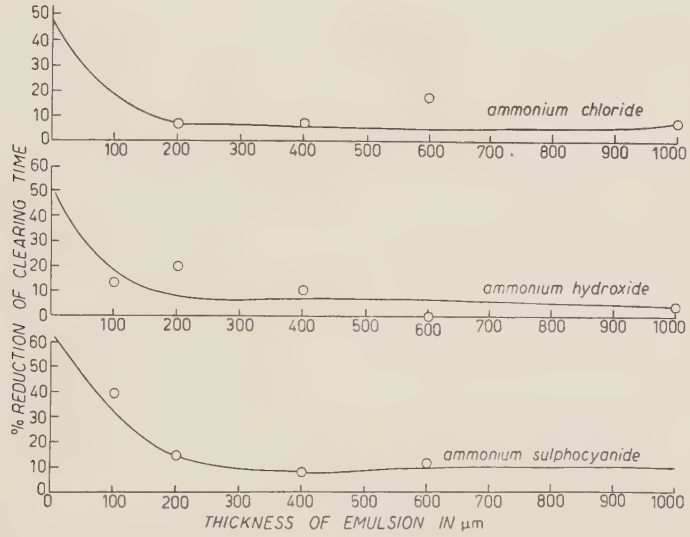

Fig. 7. – Percentage reduction in the clearing time vs. the thickness of the emulsion plates for three Ammonium salts. Thin plates have been considered to have a thickness of about 10 μm. The plotted values have been corrected for temperature variations.

percentage fall goes on decreasing in the case of Ammonium Chloride as we pass to higher thicknesses of the emulsion, while in the case of Ammonium Sulphocyanide it remains fairly constant. In the case of Ammonium Hydroxide the fall becomes very nominal at 200 μm thick plates ((5÷6)%) but after that it is fairly constant.

Similar experiments have also been performed for C2, 200 μm plates and they have also indicated exactly similar results.

3. – Discussion of the results.

It is difficult to provide an exact explanation for the nature of the curves in Fig. 2 and 3 for thin emulsions. However, the following tentative explanation can be given which can be extended for thick emulsions also. In all the curves there is a fall in the clearing time as the concentration of the Ammonium salts is increased and an optimum is indicated. The value of the optimum is dependent on the concentration of the bath (Hypo), but is independent of the nature of the Ammonium salts. The most adequate explanation can be given on the basis of the theory put forward by SHEPPARD and MEES. According to this theory, there are two factors which are responsible for a decrease in the clearing time: (i) the rapid rate of the diffusion of the fixer in the layers of the emulsion which depends on the pH of the solution, (ii) the rapid rate of the formation of the complex between Silver Halide and fixing reagent. The contribution of the first factor in the case of thin emulsion plates is very little as compared to the second factor. On the basis of this assumption, we can explain the first fall in the clearing time due to the rapid rate of complex formation; i.e. Ammonium Thiosulphate is a better complex making reagent as compared to Sodium Thiosulphate (MEES, p. 523).

As the reaction of any of the Ammonium salts with Hypo (Eqn. (1) or (2)), which goes to the formation of Ammonium Thiosulphate, is a reversible one, the forward reaction takes place only till a particular concentration of Ammonium Thiosulphate is reached. After that the dissociation of Ammonium Thiosulphate takes place and, even if higher amounts of Ammonium salts are added, its concentration does not increase. The fall in the clearing time depends on the amount of the Ammonium Thiosulphate, and when the maximum concentration is reached in a reversible reaction, the minimum is indicated in the curves of the clearing time.

The validity of this explanation is verified by its exactness in explaining the shift of the minimum as the concentration of the Hypo bath is changed (Fig. 3). The rate of a reversible reaction, will depend on the reaction velocity (say k), and if the minimum occurs when the same concentration of Ammonium Thiosulphate is reached, we can write according to the law of mass action;

e.g. for Eqn. (1),

$$(3) \qquad \frac{(\text{Mol. conc. of Amm. Chloride})^2 \times (\text{Mol. conc. of Sod. Thiosulph.})}{(\text{Mol. conc. of Amm. Thiosulph.}) \times (\text{Mol. conc. of Sod. Chloride})^2} = k\,.$$

The denominator will be a constant factor (say y) when we are considering the minimum at the same concentration of Ammonium Thiosulphate in each case, which means the same concentration of Sodium Chloride. Hence we have:

$$(4) \qquad (\text{Mol. conc. of Amm. Thiosulph.}) \times (\text{Mol. conc. of Sod. Chl.})^2 = y$$

or

$$(5) \qquad (\text{Mol. conc. of Amm. Chl.})^2 \times (\text{Mol. conc. of Sod. Thioshulph.}) = ky$$

let x, x', x'' be the quantities of Ammonium Chloride used in 10, 20 and 35 % Hypo respectively. We have for the three cases:

$$\left(\frac{x}{107}\right)^2 \left(\frac{10}{248}\right) = ky\,,$$

$$\left(\frac{x'}{107}\right)^2 \left(\frac{20}{248}\right) = ky\,, \qquad \text{or } x : x' : x'' = 1 : \frac{1}{1.4} : \frac{1}{1.8}\,.$$

$$\left(\frac{x''}{107}\right)^2 \left(\frac{35}{248}\right) = ky\,,$$

Considering x (the first point where the minimum occurs in Fig. 3*A*) as 10 % (10 g of Ammonium Chloride in 100 cm³ of Hypo solution) the value of x' and x'', i.e. the expected minimum for 20 and 35 % Hypo will occur at about 7 g in 100 cm³ (7 %) and about 5 g in the same volume (5 %), which coincides exactly with the experimental values.

Mostly the curves remain constant after this minimum value is reached because there is no opposing factor to increase it. Only in the case of Ammonium Chloride slight increase in the clearing time is indicated. This may be due to the fact that « Cl » ions have a tendency to harden the gelatin and thus the diffusion velocity of the fixer is lowered, which results as an increase in the clearing time. This hardening is effective only when the « Cl » ion concentration is sufficiently increased (i.e. at higher concentrations of Ammonium Chloride). As no such hardening agents are present in the case of Ammonium Hydroxide and Ammonium Sulphocyanide, the curves have not indicated any rise even up to a sufficiently high percentages of these salts. The Sodium Sulphocyanide (Eqn. (2)) itself acts as a complex making reagent and the curves for Ammonium Sulphocyanide show a downward trend. At this stage it is difficult to supplement any reasons for the difference in the nature of our exper-

imental curves and those given by PIPER (Fig. 1), because of the unawareness of the details and the exact circumstances under which the experiments were performed by PIPER.

In the case of thick emulsions the diffusion velocity has a more predominant role in the clearing and hence the pH of the solution becomes an important factor. We have measured the pH of the different baths for the different concentrations of the Ammonium salts and the results have been summarized in Table I.

TABLE I. – pH *of different baths having different concentrations of Ammonium salts added to* 35% *Hypo solution at* 20 °C.

Concentration of Amm. salts in %	pH		
	Ammonium Sulphocyanide	Ammonium Hydroxide	Ammonium Chloride
0.00	7.62	7.60	7.56
0.50	7.46	10.40	
1.00	7.24	10.45	7.65
1.50	7.12	—	—
2.00	6.89	—	7.12
3.00	6.86	10.60	—
4.00	6.81	—	—
5.00	6.81	10.64	6.50

The above table shows clearly that Ammonium Chloride and Ammonium Sulphocyanide baths have pH close to the value of ordinary Hypo solution. This value is close to the pH of the emulsion and other developing baths, hence no abrupt change in the pH will be found if these baths are used. Moreover the value of the pH for these baths at a concentration of $(3 \div 4)\,\%$ indicate slight acidity in their behaviour and ensures a high diffusion velocity. The situation is quite different in the case of the Ammonium Hydroxide bath where the pH ($\simeq$ 10.40) is well in the basic region and the diffusion velocity will be much lower as compared to the other two baths. That is why the curves in Fig. 5 are flatter and do not indicate any appreciable fall in the clearing time specially in the case of thick emulsions. Also the pH of this bath will be much different than the pH of the emulsion or of other developing baths and, therefore, it may not be profitable to use Ammonium Hydroxide for thick emulsions. We have observed slight colouration in 200, 400 and 600 μm plates which have been fixed in the Ammonium Hydroxide bath, although no activation on the gelatin has been indicated. Some clogging of the grains has also been observed in the plates fixed in this bath.

In the case of Ammonium Chloride a marked rise in the clearing time has been observed in thick emulsions which is unlike the thin plates. We have already given the possible reasons of this rise in clearing time based on the hardening of the gelatin by « Cl » ions which reduces the diffusion velocity of the fixer. As large volumes of gelatin are used in thick emulsions, the hardening effect becomes much more predominant as is clear by the plates of 1000 μm. The curves of Fig. 5 and 6 are much smoother.

The curves in Fig. 7 indicate that the percentage fall in the clearing time remains fairly constant from 200 μm onward. As the minimum for each thickness occurs almost at the same concentration of Ammonium salts (Fig. 4, 5 and 6), the diffusion velocity will remain the same in all the cases. In the region of thin to 200 μm the diffusion velocity decreases as the thickness increases and the percentage fall decreases. After reaching a constant value at 200 μm it becomes independent of the thicknesses of the emulsion, hence the same fall is found in the region of higher thickness (200 μm onward). It seems that two opposing factors reach a balancing point at the thickness of about 200 μm and further become independent of thickness. Of course the velocity will be different for different Ammonium salts, hence a different fall will be indicated by different baths.

4. – Effect on the grain size etc.

In order to observe the effects of the fixing baths on the developed grain size, the plates of 100, 200, 400 and 600 μm have been processed in the following manner: plates from the same batch were taken and developed with normal Amidol developer with the temperature cycle method. Four plates of the same thickness were taken in the same batch and after simultaneous treatments up to stop bath, they were transferred to four different fixing baths viz: pure Hypo, Hypo with Ammonium Hydroxide, Ammonium Chloride and Ammonium Sulphocyanide. The concentrations of the Ammonium salts were taken according to the optimum values indicated by Fig. 4, 5 and 6, for a particular thickness. The plates were left in the fixing bath for a time equal to $1\frac{1}{2}$ times the clearing time. The shrinkage factors of the different plates have been summarized in Table II, while the grain size and other relevant information are given in Table III.

The plates processed in the above described manner are found to be very nice except in the case of Ammonium Hydroxide where the plates have become slightly brownish. Except this colouration no other defect has been observed and the colouration is in no way obstructing the scanning of the plates.

From Table II it is seen that the shrinkage factors of the plates of the same

TABLE II. – *The shrinkage factors of the plates developed simultaneously but fixed in different baths.*

Thickness of the plates in μm	Shrinkage factors			
	Free Hypo	Hypo having optimum conc. of		
		Ammonium Chloride	Ammonium Hydroxide	Ammonium Sulphocyanide
100	2.00	2.26	2.08	2.00
200	2.76	2.69	2.54	2.59
400	2.44	2.46	2.40	2.40
600	2.50	2.55	2.54	2.58

batch are equal within statistical errors. This proves that the gelatin or top grains have not been completely eaten up by any of the Ammonium baths. If it would have been so the shrinkage factors should be higher in these cases. Thus Hypo bath with any of the Ammonium salts has no chemical effect on the constituents of emulsion and can be safely used for the plates up to 600 μm thickness.

TABLE III. – *Grain size at the different layers of emulsion plates fixed in different Ammonium baths.*

Thickness of plates in μm	Surface of the emulsion	Grain size in μm			
		Free Hypo	Hypo having optimum conc. of		
			Ammonium Chloride	Ammonium Hydroxide	Ammonium Sulphocyanide
200	Top	0.62	.63÷.64	.76÷.81	.62
	Middle	0.61	.56÷.65	.65÷.71	—
	Bottom	0.59	.61÷.63	.51÷.52	—
400	Top	0.67÷.69	.51÷.66	.73÷.79	.50÷.66
	Middle	0.48÷.50	.50÷.53	.73÷.76	.56÷.60
	Bottom	0.43÷.45	.47÷.51	.72÷.76	.48÷.53
600	Top	0.63÷.81	.60÷.73	.73÷.80	.62÷.65
	Middle	0.61÷.79	.46÷.51	.79÷.80	.58÷.51
	Bottom	0.54÷.70	.48÷.53	.76÷.77	.47÷.60

Table III gives the mean developed grain size for the plates. The grain sizes at the top, middle and the bottom have been measured and the range of the variation of the size has been indicated. We have thoroughly checked

the data for the whole plate. In some of the cases few very high energy tracks, in which the grain density does not vary and the grains are perfectly resolved, have been used for measuring the grain size. In some other cases the general background grains have been considered for the calculation of the grain sizes. The mean values have been shown in the Table. These results indicate that within the limits of the errors there is no variation in the grain sizes at different layers. This shows that the optimum concentration of the Ammonium salts which we have used (2% Ammonium Chloride, 1.5% Ammonium Hydroxide, and 2.5% Ammonium Sulphocyanide) does not affect the grains. This also shows that the addition of (1.5 ÷ 2)% of Ammonium Chloride to the Hypo bath does not cause any appreciable oxidation of the developed grains. Perhaps such oxidation may occur when the concentration of Ammonium Chloride is large. However, Ammonium Sulphocyanide seems to be free from any defect whatsoever.

5. – Results.

These experiments have indicated the following:

i) The addition of the Ammonium salts (Ammonium Chloride, Hydroxide and Sulphocyanide) reduces the clearing time by 50% in the case of thin emulsions.

(ii) No rise in the clearing time of thin plates has been observed at room temperature at higher concentrations of Ammonium Chloride.

(iii) The addition of these Ammonium salts reduces the clearing times of thick emulsion plates (up to 1 000 μm) by about 10% and from all considerations the addition of Ammonium Sulphocyanide is preferred as compared to the other two salts.

(iv) Ammonium Hydroxide imparts a brownish colour to the plates but does not affect the constituents of the emulsion. In these plates a clogging of the developed grains has been observed.

(v) At the optimum values, as indicated by the experimental results, the Ammonium salts do not seem to affect the grain size or the shrinkage factor of thick emulsion plates (up to 600 μm).

(vi) From pH and clarity considerations also Ammonium Sulphocyanide is highly preferred.

* * *

The authors are highly indebted to Prof. P. S. Gill, Head of the department, for his kind interest and guidance throughout the work. The help of the members of the Chemistry department of this University is also acknowledged.

RIASSUNTO (*)

Abbiamo osservato l'effètto dell'aggiunta di cloruro d'ammonio, ammoniaca e solfocianuro d'ammonio al bagno d'iposolfito di differenti concentrazioni, sul tempo di chiarificazione di emulsioni sottili e spesse (fino a 1000 μm). Nel caso di emulsioni sottili si è riscontrato un aumento per le concentrazioni superiori. Si è condotto uno studio dettagliato dei tre bagni a differenti concentrazioni di sali di ammonio e i risultati indicano un valore ottimo per particolare temperatura e spessore. Si sono anche studiati dettagliatamente gli effetti dei bagni ammonici sul calibro dei granuli sviluppati. Si sono discussi i risultati in base alla teoria prevalente. L'uso del solfocianuro di ammonio è preferibile rispetto agli altri due sali.

(*) *Traduzione a cura della Redazione.*

On the Energy Determination of the Heavy Primaries.

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(ricevuto il 28 Marzo 1958)

Summary. — The formula suggested by Peters for the energy determination of the heavy primaries, requires the knowledge of the mean energy of evaporation of the α-particles. The present work establishes that this energy depends slightly from the size of the evaporating nucleus.

1. – Introduction.

During a preceding work of our group, in which the heavy primaries of the cosmic radiation have been studied, we noticed a peculiar feature of the angular distribution of the α-particles emitted in the splittings of the heavy primaries (see: R. Cester, A. Debenedetti, C. M. Garelli, B. Quassiati, L. Tallone and M. Vigone: *Nuovo Cimento*, **7**, 371 (1958), Sect. **15**). The angle of the α-particles in the laboratory system is related to the energy of the primary. The relation normally used is the one given by Peters [1]:

$$\sqrt{\overline{\theta_\alpha^2}} = \left(\frac{\overline{E_\alpha^*} M}{3p_0^2}\right)^{\frac{1}{2}},$$

where $\overline{E_\alpha^*}$ is the mean kinetic energy of the α-particles in the rest system, M is the proton mass and p_0 is the momentum per nucleon of the incident particle. Following the information on the energy of the evaporation α-part-

(1) B. Peters: *Progress in Cosmic Rays*, **1**, 193 (1952).

icles given by LE COUTEUR and others (2), PETERS used the value $\overline{E_\alpha^*} = 12$ MeV and gave the expression

$$\sqrt{\overline{\theta_\alpha^2}} = \frac{0.06}{\varepsilon_0},$$

where θ_α is measured in radians and ε_0 is the total energy per nucleon of the incident particle in GeV.

The feature found by our group is that, in the laboratory system, the mean angle of the α-particles with the direction of the primary increases when the charge of the incident nucleus increases. The same effect has been previously found by FOWLER *et al.* (3).

The observed increase of the mean angle of the α-particles of splitting can be explained with three different hypotheses;

a) the values 0.06 for the coefficient of Peters' formula is correct for any charge of the incident nucleus and then the energy spectrum of the heavy primaries of the cosmic radiation depends on the charge of the primary;

b) the mean energy of emission of the α-particles in the rest system is not always equal to 12 MeV, but depends on the size of the nucleus from which the α-particles evaporated;

c) the mean energy of emission of the α-particles in the rest system is not always equal to 12 MeV, but depends on the type of collision in which the α-particles are emitted.

In order to decide between these hypotheses, we studied the energy of the α-particles evaporated in the stars produced by the heavy primaries. The analysis of these type of collisions (nucleus-nucleus collisions) enables us to get illuminating information; in fact:

— if hypothesis *a*) is correct we must find that the energy of emission of the α-particles of evaporation is the same for any kind of target and incident nucleus;

— if hypothesis *b*) is correct we must find a variation of the energy of emission of the α-particles of evaporation with the size of the target nucleus;

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(2) J. B. HARDING: *Phil. Mag.*, **40**, 530 (1949); K. J. LE COUTEUR: *Proc. Phys. Soc.*, A **63**, 259 (1950); D. H. PERKINS: *Proc. Roy. Soc.*, A **203**, 399 (1950); N. PAGE: *Proc. Phys. Soc.*, A **63**, 250 (1950); G. BERNARDINI, G. CORTINI and A. MANFREDINI: *Phys. Rev.*, **79**, 952 (1950); J. B. HARDING: *Phys. Mag.*, **42**, 63 (1951).

(3) P. H. FOWLER, R. R. HILLIER and C. J. WADDINGTON: *Phil. Mag.*, **2**, 293 (1957).

— if hypothesis *c*) is correct we must find a variation of the energy of emission of the α-particles of evaporation both from the type of the target nucleus as from the type of the incident nucleus.

In the last two cases it is necessary to apply a correction factor to Peters' formula in order to obtain a more accurate value of the energy of the primary.

2. – Experimental procedure and results.

235 stars produced in the emulsion by the heavy primaries ($Z \geqslant 3$) have been taken into account and the evaporation tracks of each star have been analysed. In order to distinguish the tracks of the α-particles from the ones of protons, deuterons and tritons, the most suitable measurement on tracks at the end of their range is the gap-count. A calibration on protons and α-tracks of different dip angle with the emulsion surface has been made and it has been established that the separation is possible until the dip angle is not greater than 45°. Table I collects the results of the calibration for the last 50 μm of track.

TABLE I.

dip angle	number of gaps on the last 50 μm of track	
	protons	α-particles
0° ÷ 5°	13	3
5° ÷ 22°	9	2.5
22° ÷ 35°	7	1.5
35° ÷ 45°	5	0.6

The experimental work has been done in the following way: we selected the evaporation tracks whose dip angle was $\leqslant 45°$ and on the whole range of these tracks we made the gap-count; of each track that, according to this measurement, appeared to be a track of an α-particle, we determined accurately the range and the angle with the direction of the primary. As the primaries of the stars studied in the present work have a very small dip angle with respect to the emulsion surface (the range per plate of the primaries was always greater than 3 mm), the evaporation tracks that have been analysed are more or less the ones whose vertical angle (*) with respect to the direction of the primary is $\leqslant 45°$.

(*) By « vertical angle » between two tracks, we indicate the angle between these two tracks when we project them on one plane perpendicular to the emulsion surface.

The angular distribution of the α-particles, calculated taking into account the experimental conditions above described, results to be isotropic when the primary of the star is an L nucleus ($3 \leqslant Z \leqslant 5$), while it presents a forward excess when the primary is a M nucleus ($6 \leqslant Z < 10$) or a H nucleus ($Z > 10$). The anysotropy is due to the momentum transferred to the target nucleus.

In Table II we report the mean kinetic energy $\overline{E_\alpha}$ of the α-particles in the laboratory system and the angle with the direction of the primary, $\varphi_{\frac{1}{2}}$, in which half of the tracks are contained. The energy has been calculated from the range-energy relation; the errors quoted are the statistical ones.

TABLE II.

primary	L	M	H
$\overline{E_\alpha}$	18.0 ± 1.5	20.8 ± 1.4	22.4 ± 3.3
$\varphi_{\frac{1}{2}}$	$90° \pm 7°$	$79° \pm 5°$	$66° \pm 7°$

The mean energy of the α-particles in the rest system, $\overline{E_\alpha^*}$, can be evaluated making use of the formula:

$$\beta_\alpha^* = \operatorname{tg} \varphi_{\frac{1}{2}} \sqrt{\frac{\gamma_\alpha^2 - 1}{1 + \gamma_\alpha^2 \operatorname{tg}^2 \varphi_{\frac{1}{2}}}}, \tag{1}$$

where β_α^* is the velocity of the α-particle in the rest system in velocity of light units and γ_α is the total energy of the α-particle in the laboratory system in protonic mass units. The formula (1) is valid under the assumption that the distribution of the α-particles is isotropic in the rest system of the evaporating nucleus. The results of this calculation are given in the first row of Table III.

We checked the results obtained with another method of determining the E_α^* that does not depend directly from the values of $\overline{E_\alpha}$ and $\varphi_{\frac{1}{2}}$. In fact, the momentum $\boldsymbol{P}_\alpha$ of each α-particle in the laboratory system is the composition of the momentum $\boldsymbol{P}_\alpha^*$ of the α-particle in the rest system with the momentum $\boldsymbol{T}$ transferred to the target nucleus:

$$\boldsymbol{P}_\alpha = \boldsymbol{P}_\alpha^* + \boldsymbol{T}. \tag{2}$$

The projection of this relation on the direction of the incident nucleus gives: $P_\alpha \cos\varphi = P_\alpha^* \cos\varphi^* + T$, where φ is the angle of the α-particle with the direction of the primary in the laboratory system and φ^* is the analogous angle in the rest system. Assuming that the distribution of the α-particles is iso-

tropic in the rest system, we obtain:

$$\frac{\sum P_\alpha \cos\varphi}{n_\alpha} = T \,. \tag{3}$$

Using the formulae (2) and (3) we calculate the mean value of the energy of the α-particle in the rest system. The results obtained with this method are given in the second row of Table III and are in perfect agreement with the ones obtained with the first method.

TABLE III.

primary	M	H
$\overline{E_\alpha^*}$	20.0 ± 1.0	18.0 ± 2.0
$\overline{E_\alpha^*}$	20.4 ± 1.3	20.0 ± 2.0

If we compare the figures of Table III with the value of the mean energy of the α-particles obtained for the L primary, $\overline{E_\alpha} = \overline{E_\alpha^*} = 18.0 \pm 1.5$ (see Table II), we can conclude that the mean energy of the α-particles evaporated from the target nucleus does not depend on the charge of the incident nucleus.

With the same methods above described, we calculate the mean energy of the α-particles evaporated from the nuclei of C, N, O of the emulsion and that of the α-particles evaporated from the nuclei of Ag, Br of the emulsion. For this purpose, we divided the studied stars in stars with a total number of grey and black prongs $\leqslant 7$ ($N_h \leqslant 7$) and stars with a total number of grey and black prongs > 7 ($N_h > 7$). The stars with $N_h \leqslant 7$ are due to collisions with the light elements of the emulsion and to peripheral collisions with the heavy elements. From the emulsion composition and from the evaluation of the cross-section in emulsion, the percentage of collisions with a C, N, O target is estimated to be 26%. The stars from which the α-particles are emitted, are stars with $N_h \leqslant 7$ in 36% of the cases. Consequently 28% of the stars with $N_h \leqslant 7$ represent peripheral collisions with heavy nuclei.

The results, uncorrected for peripheral collisions' contamination, are:

$\overline{E_\alpha^*} = (13.2 \pm 1.2)$ MeV for α-particles evaporated from C, N, O nuclei.

$\overline{E_\alpha^*} = (21.6 \pm 1.2)$ MeV for α-particles evaporated from Ag, Br nuclei.

If we correct these results for peripheral collisions contamination, we obtain

$\overline{E_\alpha^*} = (10.0 \pm 1.2)$ MeV for α-particles evaporated from C, N, O nuclei

$\overline{E_\alpha^*} = (21.6 \pm 1.2)$ MeV for α-particles evaporated from Ag, Br nuclei

These figures represent the lower limits for the energy of the α-particles, because they have been calculated including in the mean also the evaporation tracks with a range $\leqslant 10\ \mu$m, for which the experimental method used to distinguish the α-particle from the protons, does not give reliable results. The higher limits, obtained excluding from the calculation of the mean energy every track with a range $\leqslant 10\ \mu$m, are the following:

$\overline{E_\alpha^*} = (10.0 \pm 1.2)$ MeV for α-particles evaporated from C, N, O nuclei.

$\overline{E_\alpha^*} = (23.2 \pm 1.3)$ MeV for α-particles evaporated from Ag, Br nuclei.

Also these last figures have been corrected for the percentage of peripheral collisions.

In Table IV are summarized the lower and higher limits of the mean energy of the α-particles evaporated from nuclei of different size.

Table IV.

target nucleus	C, N, O	Ag, Br
$\overline{E_\alpha^*}$ (lower limit)	10.0 ± 1.2 (MeV)	21.6 ± 1.2 (MeV)
$\overline{E_\alpha^*}$ (higher limit)	10.0 ± 1.2 (MeV)	23.2 ± 1.3 (MeV)

In the work of our group already quoted has been given also the variation of the mean angle of the α-particles of splitting with the number of nucleon pairs involved in the collision. This variation could give an indication that the effect does not depend only on the size of the nucleus, but also on the type of collision. In the present work we find an analogous variation of the energy of the α-particles of evaporation with the number of nucleon pairs involved in the collision. The results are given in Table V.

The number of nucleon pairs depends both on the size of the incident nucleus as on the size of the target nucleus. From the results now obtained

(see Table III and Table IV) we think that the variation of the mean energy of the α-particles shown in Table V is a consequence of the variation due to the size of the nucleus, i.e., if there is a dependence of the energy of evaporation on the type of collision, this dependence is certainly much smaller than the dependence on the size of the nucleus. This fact is in agreement with the slight dependence of the mean energy of the evaporation products on the excitation energy (4).

TABLE V.

N_{np}	$\overline{E_\alpha^*}$ (MeV)
0-1-2	11.6 ± 2.0
3-4-5	16.8 ± 2.0
6-7-8-9-10-11	21.2 ± 2.0
11	22.0 ± 1.6

N_{np} indicates the number of nucleon pairs involved in the collision.

3. – Conclusions.

From the results given in the preceding section, we can conclude that hypothesis *b*) (see introduction) seems to be the correct one. In fact, the results collected in Tables III and IV indicate that, in the laboratory system, the angle of the α-particles of splitting does not depend on the size of the target nucleus, but depends on the charge of the incident nucleus.

According to the experimental data of the present work, the coefficient of Peters' formula must be taken as:

0.06 for the splitting of M nuclei,

0.08 for the splitting of H nuclei.

From the ratio of this coefficients: $0.08/0.06 = 1.33$, we would expect to find that the mean angle of the α-particles emitted in the splitting of the H nuclei is greater than the mean angle of α-particles emitted in the splittings of the M nuclei, by a factor 1.33. Actually this factor, calculated from the data of our group (given in the work already quoted), is: $(2.11 \pm 0.19)°/(1.35 \pm 0.10)° =$

(4) J. B. HARDING, S. LATTIMORE and D. H. PERKINS: *Proc. Roy. Soc.*, A **196**, 325 (1949); D. H. PERKINS: *Phil. Mag.*, **41**, 138 (1950).

$= 1.56 \pm 0.30$. The discrepancy is not large, but we think it useful to discuss a little more about this point.

The values of the mean angle of the α-particles ($\sqrt{\overline{\theta_\alpha^2}} = (2.11 \pm 0.19)°$ in the case of H primaries and $\sqrt{\overline{\theta_\alpha^2}} = (1.35 \pm 0.10)°$ in the case of M primaries) have been calculated as the root mean square of the angles of all the α-particles emitted in the studied events. Looking at Peters' formula, we see that the quantity that is proportional to the coefficient is the root mean square angle of the α-particles emitted in each single event. If we calculate (from the quoted data of our group) the mean of the root mean square angles of each event, we find:

$$\sqrt{\overline{\theta_\alpha^2}} = (1.60 \pm 0.22)° \quad \text{in the case of H primaries},$$

$$\sqrt{\overline{\theta_\alpha^2}} = (1.16 \pm 0.11)° \quad \text{in the case of M primaries}.$$

The ratio between these two angles is: 1.38 ± 0.36.

The difference between the two methods of calculating a mean value of the angle of the α-particles is due to two facts: first, in one method we make an arithmetic mean and in the other method we make a root mean square mean; second, in one method each event has the same statistical weight, whereas in the other method the events have a statistical weight increasing with the increase of the number of α-particles emitted in the splitting. The arithmetic mean is always lower than the root mean square one; the difference in the statistical weights has an influence on the final result only if there is a variation of the angle of the α-particles that depends on the number of α-particles emitted in the splitting. This variation has been found by FOWLER *et al.* [3] and it is an important point to be considered if we want to understand the very high values of the evaporation energy calculated for the α-particles by these authors.

At the present point of our knowledge it is not possible to say which is the physical meaning of the variation of the angle of the α-particles with the number of α-particles emitted from the same nucleus and which is the phenomenon that produces it. It would be useful to know the energy of evaporation of the α-particles in the rest system when one α-particle only and when more α-particles are evaporated. Unfortunately we cannot use the experimental results of the present work for this purpose, because, having analysed only the evaporation tracks that make a vertical angle $\leqslant 45°$ with the direction of the primary, we do not know the number of α-particles evaporated in each star.

The result of the present work clears up the question about the validity of Peters' formula. Actually the value 0.06 of the coefficient, choosen by PETERS, is the best one for the medium elements, while for the heavy ones a value of 0.08 seems more correct. This change in the value of the coeffi-

cient takes into account the variation of the mean energy of evaporation of the α-particles with the size of the nucleus that was already pointed out by PERKINS [5]. The mean energies of evaporation found by us are only slightly higher than the ones given by PERKINS, but not so high as the ones suggested by FOWLER *et al.* [3].

* * *

We wish to express our thanks to Drs. A. DEBENEDETTI, R. CESTER, L. TALLONE and V. BISI, for many useful discussions.

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[5] D. H. PERKINS: *Phil. Mag.*, **40**, 601 (1949).

RIASSUNTO

La formula proposta da PETERS per la determinazione dell'energia dei primari pesanti, richiede la conoscenza dell'energia media delle particelle α di evaporazione Il presente lavoro prova che questa energia dipende leggermente dalla dimensione del nucleo che evapora.

Plasma Loss from Magnetic Bottles.

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(ricevuto il 28 Marzo 1958)

Summary. — The various shapes of magnetic fields devised in order to confine a plasma (magnetic bottles) let always leak out part of it, for different reasons. One of the most important of these losses, is that due to particles gliding along the lines of magnetic force. In this paper a general method is proposed for evaluating approximately this loss, and it is applied to two particular kinds of magnetic bottles.

Introduction.

It is generally appreciated that the practical utilization of power from thermonuclear reactions depends on the perfection of insulation of the thermonuclear fuel from all sinks of thermal energy ([1,2]). The most efficient method of achieving this is to confine the fuel, which at the temperatures suitable for fusion reactions appears in the form of a fully ionized gas, by strong magnetic fields. These magnetic fields can be generated either by currents in the ionized gas itself ([3]), or by currents in conductors outside the gas ([4]) or by both ([5]).

([1]) R. F. Post: *Rev. Mod. Phys.*, **28**, 338 (1956).

([2]) J. G. Linhart: *Nuclear Engineering* (Feb. 1957), p. 60.

([3]) P. C. Thonemann, E. P. Butt, R. Carruthers, A. N. Dellis, D. W. Fry, A. Gibson, G. N. Harding, D. J. Lees, R. P. Mc Whirter, R. S. Pease, S. A. Ramsden and S. Ward: *Nature*, **181**, 217 (1958).

([4]) L. Spitzer: *Nature*, **181**, 221 (1958).

([5]) S. J. Roberts and R. J. Tayler: AERE rep. T/R 2138; AERE rep. T/R 2264.

In this paper, we shall be concerned with the confinement of plasma by magnetic fields generated by currents outside the plasma.

Such a magnetic field configuration is often called a « magnetic bottle ». The loss of heat energy from plasma confined in such a bottle is mainly due to

a) particle losses,

b) bremsstrahlen.

The loss *b*) has been discussed elsewhere (6) and does not depend on the form of the magnetic bottle.

The loss *a*) can be divided into two categories:

α) particle loss due to plasma instabilities,

β) particle loss due to collisions between particles leading to plasma diffusion out of the magnetic bottle.

We shall concentrate mainly on the diffusion loss through the holes in magnetic bottles. We shall first develop an approximate theory and apply it to two particularly simple magnetic bottles. In the second part of the paper we shall develop a more general formula for the particle loss from the ends of any flux tube in a magnetic bottle (*). Subsequently the results will be discussed in connection with thermonuclear applications.

1. – Let us consider a simple magnetic bottle formed by coaxial magnetic mirrors facing each other (Fig. 1) and containing hydrogen plasma. Let us assume that at time $t = 0$ the velocity distribution of both electrons and protons is maxwellian with a temperature T. It can be shown that as the magnetic moment of each particle is invariant between two successive collisions

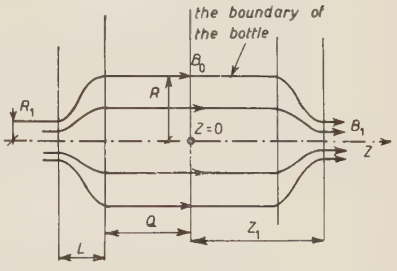

Fig. 1.

$$\frac{\sin^2\theta}{B} = \text{const}\,, \tag{1}$$

where θ is the angle between the velocity vector $\boldsymbol{v}$ and the $\boldsymbol{B}$ line belonging to the guiding centre of the particle.

(6) W. Heitler: *Quantum theory of radiation* (Oxford, 1954).

(*) The diffusion loss at right angles to the walls of a flux-tube is due mainly to electron-positive ion collisions and has been discussed elsewhere (7).

(7) L. Spitzer: *Physics of fully ionized gases* (New York, 1956), p. 36.

In absence of collisions, particles for which

$$(\sin^2\theta)_{z=0} < \frac{B(z=0)}{B(z=z_1)} = b\,, \tag{2}$$

will leak through the magnetic mirrors. Those for which

$$1 > (\sin^2\theta)_{z=0} > \sin^2\alpha = b \tag{3}$$

will remain within the bottle.

One can, therefore, consider the plasma between two magnetic mirrors as a mixture of two gases. The first, say gas A, in which all the particles obey eq. (3) and are, therefore, fully confined. The second, gas B, in which particles obey eq. (2) and therefore, are not confined.

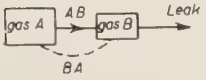

Fig. 2.

If collisions between particles occur, then the loss from the magnetic bottle containing a mixture of A and B is determined by the magnitude of collision-induced transitions from A to B and B to A (Fig. 2). This interchange of particles between the gases A and B can be represented by a diagram in the velocity space (Fig. 3).

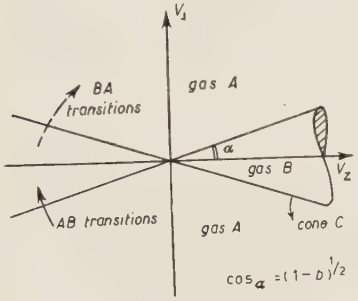

Fig. 3.

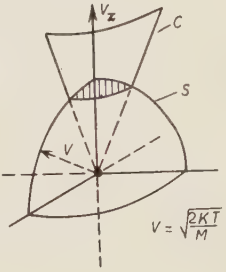

Fig. 4.

If the apex angle α is small then the probability that a collision between two protons belonging to the gas A will result in transfer of one of them to the gas B is approximately (Fig. 4)

$$\omega = \frac{\text{surface of spherical sectors cut out by the cone } C}{\text{surface of a sphere } S} = \frac{1}{2}\sin^2\alpha\,,$$

and using equation (3) one has

$$\omega = \tfrac{1}{2}b\,. \tag{4}$$

The factor b may be evaluated as follows (see Fig. 1):

$$b = \frac{B_0}{B_1} = \left(\frac{R_1}{R}\right)^2 .$$

The transfer AB occurs, therefore, at a rate

$$N_{AB} = \frac{n_A}{\tau}\,\omega \ (\text{part/cm}^3, \text{s}) , \tag{5}$$

where n_A is the density of the gas A and τ is the corresponding mean collision time.

The gas A is, of course, composed of equal number of electrons and protons and we should, therefore, work out separately

$$N_{AB}^- \qquad \text{and} \qquad N_{AB}^+ .$$

Eq. (5) suggests, that the loss of protons from A is smaller than that of electrons. However, the actual loss of plasma from the magnetic bottle is determined by the loss of protons. The electrons are then held by the electric field of the protons. Thus τ_A can be taken as the mean time between proton-proton collisions and this is

$$\tau = \frac{11.4}{\ln \Lambda}\,\frac{T^{\frac{3}{2}}}{n} \simeq \frac{T^{\frac{3}{2}}}{n} . \tag{6}$$

With this eq. (5) becomes

$$N_{AB} = \frac{1}{2}\,\frac{n_A n}{T^{\frac{3}{2}}}\cdot b \ (\text{part/cm}^3, \text{s}) . \tag{7}$$

Also with our assumption that $\alpha \ll 1$ one has $n_A \gg n_B$ ans therefore $n_A \sim n$. Thus

$$N_{AB} = \frac{1}{2}\,\frac{n^2 b}{T^{\frac{3}{2}}} \ (\text{part/cm}^3, \text{s}) . \tag{7a}$$

The transfer BA occurs at a rate

$$N_{AB} = \frac{n_B}{\tau'} , \tag{8}$$

where τ' is the time between two collisions at an angle larger than α. This is

$$\tau' = \frac{(Mv^2)^2}{nv\pi e^4 \ln \Lambda} \sin^2 \alpha/2 = \frac{1}{2}\, b\, \frac{(kT)^2}{\pi n e^4 \sqrt{2kT/M} \ln \Lambda} \sim \frac{b}{8}\,\frac{T^{\frac{3}{2}}}{n} .$$

Therefore eq. (8) becomes

$$N_{BA} = \frac{8 n_B n}{b T^{\frac{3}{2}}} \text{ (part/cm}^3\text{, s)} . \tag{8a}$$

We shall assume that (*)

$$\frac{n_B}{n} < \frac{b^2}{16} , \tag{9}$$

and neglect N_{BA}. This could be justified also on the grounds that we have neglected small angle collisions in evaluating N_{AB}.

The loss of plasma is then given by eq. (7*a*).

The corresponding loss of energy is

$$W = \frac{3}{2} kT \int_{\Omega} N_{AB} \, \mathrm{d}\Omega \text{ (erg/s)} , \tag{10}$$

where Ω is the volume of the bottle.

An average energy loss per cm³ is

$$W_0 = \frac{3}{4} k \frac{n^2}{T^{\frac{1}{2}}} b \text{ (erg/cm}^3\text{, s)} , \tag{10a}$$

and using the definition of b in terms of bottle-geometry one has

$$W_0 = \frac{3}{4} k \frac{n^2}{T^{\frac{1}{2}}} \left(\frac{R_1}{R}\right)^2 \text{ (erg/cm}^3\text{, s)} . \tag{10b}$$

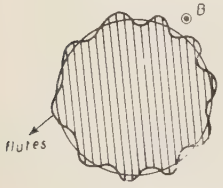

Fig. 5.

The diffusion loss from the simple bottle discussed so far may be expected to be inferior to the loss due to flute-instability. This instability (Fig. 5) will occur in magnetic bottles in which [8]

$$\int_s \frac{\mathrm{d}s}{R_c r B^2} > 0 , \tag{11}$$

(*) Even if the density in the velocity space remained spherically symmetrical

$$\frac{n_B}{n} = \frac{\frac{2}{3}\pi \frac{1}{2} b}{\frac{4}{3}\pi} = \frac{1}{4} b .$$

In neglecting N_{BA} we assume that inside the cone C the density of particles has dropped by a factor of at least $\frac{1}{4}b$ below the mean density in the sphere S. This is not always true (see eq. (35)).

[8] M. N. ROSENBLUTH and C. L. LONGMIRE: *Ann. of Phys.*, **1**, 120 (1957).

where the integration is along a field line through the bottle and R_c is the radius of curvature of the flux line.

An example of magnetic bottle which, according to this criterion, should not exhibit a flute instability is a quadrupole magnetic field (Fig. 6). The diffusion loss should then be the only appreciable particle loss from a plasma confined in such a bottle.

There are three holes in the quadrupole bottle. It is obvious that the large aperture of the central cylindrical gap G will permit the largest leak. One can again solve the problem of loss by the approximate method of the two-gas model.

The allocation of velocity space to the gases A and B is asymmetric if $B_2 \neq B_1$. It can be shown that

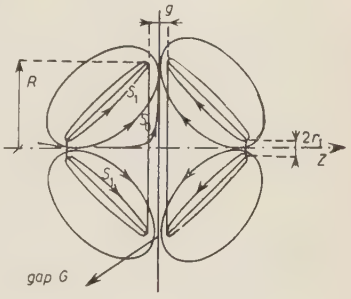

Fig. 6.

$$\frac{B_1}{B_2} = \frac{Rg}{r_1^2} . \tag{12}$$

where B_1 respectively B_2 are the field strengths in the axial aperture and in the gap respectively.

In order that $B_1 = B_2$ one must make

$$g = r_0 \left(\frac{r_0}{R}\right), \tag{12a}$$

which encounters technological difficulties if $R/r_0 \gg 1$.

Also the apex angle of the cone C dividing the velocity space changes from zero for the rectangular field-line S_0 to some finite value for the field line S_1 tangential to the coils. Along any flux-tube between S_0 and S_1 one encounters a field minimum $B_{\min}$. An approximate expression for an average value for $B_{\min}$ is

$$\overline{B}_{\min} = \frac{\pi r_0^2 B_1}{\pi (R/2)^2} = 4 \left(\frac{r_0}{R}\right)^2 B_1 , \tag{13}$$

and we get for

$$\bar{b} = \frac{\overline{B}_{\min}}{B_2} = 4 \left(\frac{r_0}{R}\right)^2 \frac{Rg}{r_0^2} = 4 \frac{g}{R} . \tag{14}$$

With this the loss of energy from a cm³ of the confined plasma is found in the same way as in the case of a simple bottle and it is

$$W_0 = 3k \frac{n^2}{T^{\frac{1}{2}}} \frac{g}{R} \text{ (erg/cm}^3\text{, s)} . \tag{15}$$

2. – In this section more general formulae for the particle loss from a magnetic flux tube will be derived.

Let us consider a plasma in a steady magnetic field sufficiently strong to legitimate the assumption that the particles move by spiralling along the lines of force. The movement of each particle (or better of its « guiding center ») can then be treated as a one-dimensional problem on a given trajectory. Let us call s the curvilinear abscissa along the line of force, $\boldsymbol{v}$ the velocity and θ its angle with $\boldsymbol{B}$.

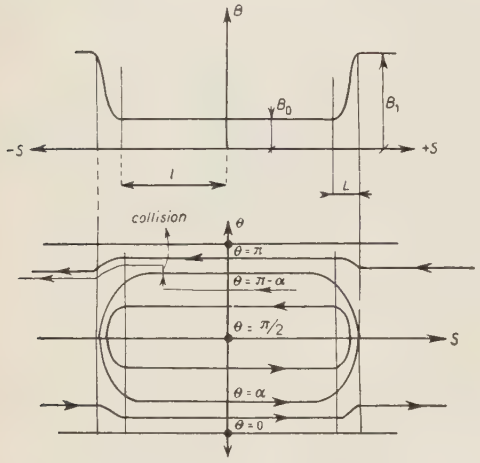

Fig. 7.

In a plane s, θ (Fig. 7), each particle will be represented by a point moving along a line represented by eq. (1), with a velocity whose s component is $v \cos\theta$.

Let us simplify the problem by assuming that all particles possess the same magnitude of velocity $|v|$. This assumption is in contradiction with the existence of collisions, but this does not affect very much the results.

We shall formulate the equation of balance for representative points in a small element of area $[\mathrm{d}s\,\mathrm{d}\theta]$. Let

$f(s, \theta, t)$ be the density of representative points in the plane s, θ at time t in a unit flux tube.

$F(s, t) = \int_0^{\pi} f(s, \theta, t)\,\mathrm{d}\theta$ the linear density of particles in a unit flux tube.

$Q(s, t)\,\mathrm{d}s\,\mathrm{d}t$ the number of particles (of each sign) produced in $\mathrm{d}s$ in time $\mathrm{d}t$.

Let us assume that they are produced with isotropic distribution of velocities. Thus the number of representative points produced in the element $\mathrm{d}s\,\mathrm{d}\theta$ in time $\mathrm{d}t$ is

$$\tfrac{1}{2} Q(s, t) \sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}t\,.$$

Let $\nu(s, t)$ be the collision frequency and suppose that each collision turns $\boldsymbol{v}$ in a new direction completely at random, so that the number of points entering

in $[\mathrm{d}s \cdot \mathrm{d}\theta]$ because of collisions in time $\mathrm{d}t$ is

$$\frac{\nu}{2} F \sin\theta \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t \, ,$$

while those leaving amount to $\nu f \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t$ (*).

Making use of eq. (1) it is easily seen that the balance equation for any flux tube is (writing B' for $\mathrm{d}B/\mathrm{d}s$)

$$\frac{\partial f}{\partial t} = \frac{Q}{2} \sin\theta - \frac{\partial f}{\partial s} v \cos\theta - \frac{\partial}{\partial \theta} (f \sin\theta) \frac{B' v}{2B} + \frac{\nu}{2} F \sin\theta - \nu f \, . \tag{16}$$

General solution for the stationary state. – If everything is independent of t, we can write eq. (16) as follows

$$\frac{\partial f}{\partial s} v \cos\theta + \frac{\partial f}{\partial \theta} \frac{B' v}{2B} \sin\theta = \frac{Q}{2} \sin\theta - \frac{B' v}{2B} f \cos\theta + \frac{\nu}{2} F \sin\theta - \nu f \, . \tag{17}$$

If one, for the moment, considers F as a given function, one can treat eq. (17) as a linear diff. eq. of the first order and integrate it formally by the method of characteristics.

The differential system for the characteristic curves of eq. (17) is

$$\frac{\mathrm{d}s}{v \cos\theta} = \frac{\mathrm{d}\theta}{(B' v/2B) \sin\theta} = \frac{\mathrm{d}f}{(Q/2) \sin\theta - (B' v/2B) f \cos\theta + \nu \{(F/2) \sin\theta - f\}} \, . \tag{18}$$

The first of these equations can be integrated immediately and gives

$$\frac{\sqrt{B}}{\sin\theta} = C \, , \tag{19}$$

which is identical with (1); thus the characteristic curves in the $s - \theta$ plane coincide, between collisions, with the trajectories of the representative points.

The second equation can be written (utilizing eq. (19) for eliminating θ)

$$\frac{\partial f}{\partial s} + \left(\frac{B'}{2B} \pm \frac{\nu}{v} \frac{C}{\sqrt{C^2 - B}} \right) f = \pm \frac{1}{2v} (Q + \nu F) \sqrt{\frac{B}{C^2 - B}} \, , \tag{20}$$

where the upper sign holds for $\theta < \pi/2$, the lower one for $\theta > \pi/2$.

(*) In a more refined theory ν would have to involve also small angle scattering.

Putting

$$I(C, s) = \sqrt{B} \exp\left[\pm \int_0^s \frac{Cv\,\mathrm{d}s'}{v\sqrt{C^2 - B(s')}}\right], \tag{21}$$

the general solution of eq. (20) is

$$f = \pm \frac{1}{2vI(C, s)} \int_0^s I(C, s')[Q(s') + vF(s')] \sqrt{\frac{B(s')}{C^2 - B(s')}}\,\mathrm{d}s' + \frac{C'}{I(C, s)}\,. \tag{22}$$

The general solution of eq. (17) is obtained by putting in eq. (22) $C' = \psi(C)$ where ψ is an arbitrary function, and then substituting everywhere for C the expression (19). Thus we have

$$f(s, \theta) = \pm \frac{1}{2vI(C, s)} \int_0^s I(C, s')[Q(s') + vF(s')] \sqrt{\frac{B(s')}{C^2 - B(s')}}\,\mathrm{d}s' + \frac{\psi(C)}{I(C, s)}\,. \tag{23}$$

The function $\psi(C)$ must be determined by means of the boundary conditions. Finally, integrating eq. (23) with respect to θ between 0 and π, one gets an integral equation for F.

This general procedure will be, however, usually too complicated to be carried out without some drastic approximations.

Application to the confinement by a longitudinal magnetic field. – Suppose now that the plasma is confined in a cylindrical vessel by a longitudinal magnetic field, approximately uniform (B_0) in the central part (for $l < s < l$) but increasing towards the ends so as to give rise to two magnetic mirrors (Fig. 1). Let us suppose that the rise from the value B_0 to the maximum B_1 takes place in a length L short compared to l, but still large compared with the radius of the spiral, i.e.

$$f \gg L \gg \frac{m_0 v}{eB}\,.$$

The trajectories of the representative points in the $s - \theta$ plane, given by eq. (19) are shown in Fig. 7. Those with $C < \sqrt{B_1}$ will be closed curves and represent trapped particles, while those with $C > \sqrt{B_1}$ will have two open branches, the one for $0 < \theta < \pi/2$, the other for $\pi/2 < \theta < \pi$, and represent particles entering at one end of the bottle and leaving through the other.

Each representative point will move along one of these lines (in the direction of the arrows) and, occasionally, jump to another because of a col-

lision. The boundary between open and closed curves is given, in the central section, by the ordinates θ_0 and $\pi - \theta_0$, with

$$\sin \alpha = \sqrt{\frac{B_0}{B_1}} = \sqrt{b} \, .$$

Let us now apply formula (23) to the particles spiralling on one of the flux tubes of this field. Let us introduce the concept of mean free path $\lambda = v/\nu$

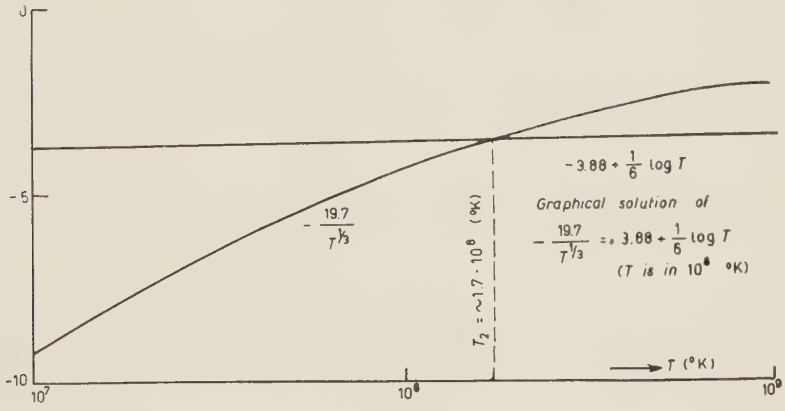

Fig. 8.

and assume that, between $-l$ and l, Q and F can be considered constants. Then, for this interval, equation (21) gives

$$I = \sqrt{B_0} \exp\left[\frac{s'}{\lambda \cos\theta}\right]. \tag{24}$$

Putting this into eq. (23) we get, for the same interval

$$f = \frac{\operatorname{tg}\theta}{2v} \exp\left[-\frac{s}{\lambda \cos\theta}\right] \int_0^s [Q + \nu F(s')] \exp\left[\frac{s'}{\lambda \cos\theta}\right] \mathrm{d}s' + \\ + \frac{1}{\sqrt{B_0}} \exp\left[-\frac{s}{\lambda \cos\theta}\right] \psi\left(\frac{\sqrt{B_0}}{\sin\theta}\right), \tag{25}$$

or, denoting by $\overline{Q} + \overline{F\nu}$ a convenient average

$$f = \frac{\lambda}{2v} (\overline{Q} + \overline{\nu F}) \left(1 - \exp\left[-\frac{s}{\lambda \cos\theta}\right]\right) \sin\theta + \frac{1}{\sqrt{B_0}} \exp\left[-\frac{s}{\lambda \cos\theta}\right] \psi\left(\frac{\sqrt{B_0}}{\sin\theta}\right). \tag{26}$$

We shall now take into account the condition that no particles enter from the ends, that is from the points $s = \pm (l+L)$. This condition can be replaced by an approximate one, namely that no particles enter from the points $s = \pm l$ except those which are coming back from a reflection on the magnetic mirror: this amounts to neglecting the particles generated in the terminal sections of length L. Therefore we impose the condition that $f = 0$ for $s = -l$, $\theta < \alpha$ and for $s = l$, $\theta > \pi - \alpha$.

This condition determines ψ (for $\sin\theta < \sqrt{b}$) and eq. (26) becomes

$$f = \frac{1}{2v}(\overline{Q} + \overline{vF})\left(1 - \exp\left[-\frac{s+l}{\lambda\cos\theta}\right]\right)\sin\theta \qquad \text{for} \qquad \begin{pmatrix} -l < s < +l \\ \sin\theta < \sqrt{b} \end{pmatrix}. \tag{27}$$

The number φ of particles escaping from each end per unit time is given by

$$\varphi = \int_0^{\pi} vf(l,\theta)\cos\theta\,\mathrm{d}\theta = \frac{\lambda}{2}(\overline{Q} + \overline{vF})\int_0^{\pi}\left(1 - \exp\left[-\frac{2s_m}{\lambda\cos\theta}\right]\right)\sin\theta\cos\theta\,\mathrm{d}\theta\,. \tag{28}$$

The losses Φ per unit volume from the whole vessel (from both ends) are related to φ as follows. Let S be the cross-section of the vessel, n the volume density of particles. As there are B unit flux tubes per unit cross-section, i.e. $\overline{F} = n/B_0$, one obtains

$$\Phi = \frac{2\varphi S B_0}{2lS} = \frac{B_0}{l}\varphi\,,$$

and using eq. (28)

$$\Phi = \frac{\lambda}{2}(\overline{Q}^* + \overline{vn})\int_0^{\pi}\left(1 - \exp\left[-\frac{2l}{\lambda\cos\theta}\right]\right)\sin\theta\cos\theta\,\mathrm{d}\theta\,, \tag{29}$$

(denoting by $Q^* = QB_0$ the density of the source per unit volume).

Let us calculate the integral for two limiting cases.

Case a). $\qquad \lambda \gg 2l\,, \qquad B_0 \ll B_1\,.$

In this case the exponential term in eq. (29) may be replaced by $1 - (2l/\lambda\cos\theta)$ and the integral becomes

$$\frac{2l}{\lambda}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = \frac{2l}{\lambda}(1 - \cos\alpha)\,.$$

As

$$\alpha \ll 1\,, \qquad 1 - \cos\alpha \simeq \frac{\alpha^2}{2} \simeq \frac{b}{2}\,,$$

eq. (29) becomes

$$\Phi \cong \tfrac{1}{2}(\overline{Q}^* + \overline{vn})b \,. \tag{30}$$

In the stationary state, the internal density n will automatically adjust itself to such a value, that the escaping particles just counterbalance those produced by the source, that is $\overline{Q} = \Phi$. Therefore

$$\Phi\left(1 - \frac{b}{2}\right) = \frac{1}{2}\,\overline{nv}b \,, \tag{31}$$

and neglecting $b/2$ in comparison with 1 we have

$$\Phi = \sim \frac{b}{2}\,\overline{nv} \,. \tag{32}$$

This is the same as eq. (7*a*) obtained in Sect. **1**.

Case b). $\qquad \lambda \ll 2l \qquad$ and $\qquad B_0 \ll B_1 \,.$

In this case the exponential term in eq. (28) may be neglected and there remains

$$\Phi = \frac{\lambda}{2l}\,(\overline{Q}^* + \overline{nv})\,\frac{b}{2} \,. \tag{33}$$

As mentionend above, $Q^* = \Phi$ and one gets

$$\Phi\left(1 - \frac{b}{4}\,\frac{\lambda}{l}\right) = \frac{\lambda}{4}\,\frac{\overline{nv}}{l} \,, \tag{34}$$

or approximately

$$\Phi \simeq \frac{n}{4}\,\frac{v}{l}\,b \,. \tag{35}$$

This result can be justified intuitively using the model of two gases.

Gas A is formed by particles with $\sin \alpha^2 > b$, gas B by particles with $\sin \alpha^2 < b$.

If $\lambda \ll 2l$, each B particle collides many times before reaching the end, and has many opportunities to become an A particle: thus it is no more legitimate to neglect $B \to A$ transitions. It will be more correct to admit that they compensate (approximately) the $A \to B$ transitions.

The loss of particles is in this case determined not by the rate of formation of gas B, but by its velocity of displacement towards the ends, which

is $v\cos\theta \simeq v$. The loss from each end is (for unit time and unit area) $n_B v/2$, and for unit volume and both ends $(n_B/2)(v/l) = \omega(n/2)(v/l)$. This agrees with eq. (35).

Non stationary problems can be treated along the same line, starting from eq. (16). The general solution has of course, two arbitrary functions, one of which must be determined by the initial conditions.

3. – Conclusions.

In order that the magnetic bottles described here could be used to confine plasma in thermonuclear reactors, the energy loss given by equations (10*b*) and (15) must be appreciably less than the nuclear output ([2]). For the TD reaction running at a temperature $T \sim 10^9$ (°K) one has for the output power density ([1])

$$W_N = \tfrac{1}{2} n^2 \langle \sigma v_{\mathrm{DT}} \rangle_{av} \cdot \varepsilon \ \ (\mathrm{erg/cm^3, s}), \tag{36}$$

where

$$\langle \sigma v_{\mathrm{DT}} \rangle_{av} \sim 10^{-15} \quad (\mathrm{cm^3/s})$$

$$\varepsilon \sim \tfrac{1}{4} \cdot 10^{-4} \ (\mathrm{erg}).$$

Therefore

$$W_N = \tfrac{1}{8} \cdot n^2 \cdot 10^{-19} \quad (\mathrm{erg/cm^3, s}). \tag{36a}$$

Let us form a ratio between the nuclear output density and the energy loss density for the quadrupole bottle of Fig. 6. Thus

$$\frac{W_N}{W_0} = \eta \,. \tag{37}$$

Using eq. (15) one gets

$$\eta = \frac{10^{-19}}{24\,K} T^{\frac{1}{2}} \frac{R}{g} \sim \frac{R}{g} \,. \tag{38}$$

Thus at these very high temperatures the energy loss due to particle diffusion is small compared with the nuclear output. However, as the plasma temperature decreases, the W_N decreases rapidly, whereas W_0 slowly increases. The critical temperature T_c is obtained from the reactor equation ([2]) which in our case can be written as

$$\eta\left(T, \frac{R}{g}\right) \geqslant 3 \,.$$

This gives for $R/g \sim 100$ a critical temperature

$$T_c \sim 1.7 \cdot 10^8 \ (^\circ\mathrm{K}),$$

which is about 3 times higher than that obtained by considering the bremsstrahlen loss alone.

It thus appears that magnetic bottles with holes can be considered as means of plasma confinement for thermonuclear processes, provided the plasma confinement is a stable one.

RIASSUNTO

Le varie forme di campo magnetico immaginate per imprigionare un plasma (bottiglie magnetiche) ne lasciano sempre sfuggire una parte, per diverse ragioni. Una delle più importanti fughe è quella dovuta alle particelle che scorrono lungo le linee di forza magnetica. Nel presente lavoro si propone un metodo generale per valutare approssimativamente questa perdita, e lo si applica a due forme particolari di bottiglia magnetica.

On the Energy and Pulse Spectra of a Fast Ionization Chamber.

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(ricevuto il 6 Marzo 1958)

Summary. — An analytical investigation is carried out on the influence of geometrical factors of a pulse ionization chamber on the correspondence between energy and pulse spectra. General formulas are given for plane, cylindrical and spherical chambers. The method is suitable to be handled in practical cases and examples are given.

The energy distribution of the ionization events detected by a pulse ionization chamber can be derived from the pulse height distribution. In general, the energy and pulse height distributions have different shapes for two main reasons, *a*) the chamber geometry and *b*) the range of the ionizing particles is not negligible compared with the dimensions of the chamber.

It is the purpose of this note to give an analytical treatment of the dependence on the chamber geometry of the correspondance between the energy distribution and the pulse height distribution, under the hypothesis that the detected events are caused by ionizing particles of infinitesimal range. This assumption is acceptable, for instance, in order to describe the gross features of ionization events produced either by recoil protons, arising from neutron bombardment, or by nuclear disintegrations, provided the gas pressure be sufficiently high and the chamber dimensions large enough. In both cases, a certain number of ion pairs is formed within an infinitesimal volume which will be not assumed to be preferentially located within the sensitive chamber space. Furthermore, we assume the number of the collected electrons to be proportional to the energy delivered by the ionization event, and therefore

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all effects due to attachment, recombination and diffusion processes will be ignored.

Under these hypotheses, let us suppose that the energy distribution of the ionization events, detected by the pulse chamber during a certain time interval, is expressed by the general law

1) $$N(E)\,\mathrm{d}E\,,$$

where $N(E)$ is the number of events per energy interval at energy E, varying within the range

(2) $$E_1 \leqslant E \leqslant E_2\,.$$

Since the total number of the detected events is given by

(3) $$\mathcal{N} = \int_{E_2}^{E_1} N(E)\,\mathrm{d}E\,,$$

the probability $p(E)\mathrm{d}E$ that the energy of an event lies between E and $E+dE$ reads

(4) $$p(E)\mathrm{d}E = \frac{N(E)}{\mathcal{N}}\,\mathrm{d}E\,.$$

The function $p(E)$, defined in Eq. (4), varies between zero and 1 for E varying over the range (2), and is equal to zero for $E < E_1$ and $E > E_2$.

The height P of a pulse depends both on the energy E of the ionization event and on the position r where the event has taken place. The relation between P, E and r for the ionization chambers generally used is given by:

(5-a) plane chamber $$P = k_1 E \frac{r-a}{b-a}\,,$$

(5-b) cylindrical chamber $$P = k_2 E \frac{\ln r/a}{\ln b/a}\,,$$

(5-c) spherical chamber $$P = k_3 E \frac{1-a/r}{1-a/b}\,.$$

In Eqs. (5-b) and (5-c), b and a are the radii of the outer and the collecting electrodes respectively; in Eq. (5-a), b and a represent the distances of the outer and the collecting electrodes respectively, from a reference plane parallel to the electrodes. In all cases, the value $r = a$ corresponds to the collecting electrode. The constants k_ν ($\nu = 1, 2, 3$) depend on the gas filling, on the electrostatic characteristics of the chamber and on the gain of the amplifier, which is supposed to be linear. These constants are determined through ca-

libration of the chamber by means of a monoenergetic source. We can write Eqs. (5-*a*), (5-*b*) and (5-*c*) in the general form

$$P = E f_\nu(r) , \tag{6}$$

where $\nu = 1, 2, 3$ for the plane, cylindrical and spherical chambers respectively. In the three cases $f_\nu(r)$ is written as follows:

$$f_1(r) = k_1 \frac{r - a}{b - a} , \tag{7-a}$$

$$f_2(r) = k_2 \frac{\ln r/a}{\ln b/a} , \tag{7-b}$$

$$f_3(r) = k_3 \frac{1 - a/r}{1 - a/b} . \tag{7-c}$$

The function $f_\nu(r)$ is easily seen to vary from zero to k_ν for r varying from a to b. Plotting in the plane (P, E) the function $P(E, r)$ defined in Eq. (6) one obtaines a family of straight lines of angular coefficient $f_\nu(r)$. The values $P_1 = k_\nu E_1$ and $P_2 = k_\nu E_2$ are the maximum pulse heights produced by the events of minimum and maximum energy respectively. The position r where an event of energy E must occur in order to produce a pulse of height P is derived from the value of the angular coefficient of the straight line passing through the point (P, E), through the equation

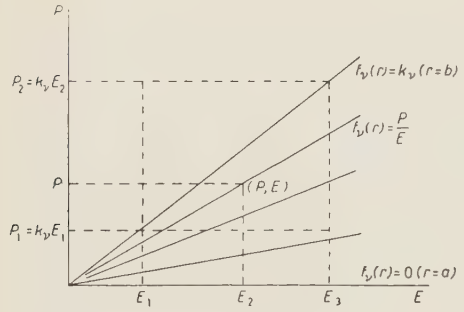

Fig. 1. – Graph of P of Eq. (6) against energy E for different values of the angular coefficient $f_\nu(r)$.

$$r = f_\nu^{-1}\left(\frac{P}{E}\right) . \tag{8}$$

Let us now consider a pulse lying between P and $P + \mathrm{d}P$. This pulse P can be produced by an event occurring at a certain position r provided the energy of the event lies between

$$E(P, r) \quad \text{and} \quad E(P, r) + \frac{\partial E}{\partial P}\mathrm{d}P , \tag{9}$$

that is, taking into account Eq. (6),

$$P[f_\nu(r)]^{-1} \quad \text{and} \quad P[f_\nu(r)]^{-1} + [f_\nu(r)]^{-1}\mathrm{d}P . \tag{10}$$

From Eq. (4) is found the probability for an event to have an energy lying into the interval (10), i.e.,

$$p(E)\,\mathrm{d}E = \frac{N[E(P,r)]}{\mathcal{N}}\,[f_\nu(r)]^{-1}\,\mathrm{d}P\,. \tag{11}$$

This event may occur anywhere into the sensitive volume of the chamber, and the probability $g(r)\,\mathrm{d}r$ for it to occur at the position r may be written in the general form

$$g(r)\,\mathrm{d}r = \frac{\nu r^{\nu-1}}{b^\nu - a^\nu}\,\mathrm{d}r\,, \tag{12}$$

where, as previously stated, $\nu = 1, 2, 3$ for the plane, cylindrical and spherical chambers respectively.

Using the preceding results, we can finally define the probability $p(P)\,\mathrm{d}P$ that a pulse has a height lying between P and $P+\mathrm{d}P$. As a preliminary step, we evaluate the probability that an event occurs at r and produces a pulse P. This is obtained by multiplying Eqs. (11) and (12), i.e.,

$$p(E)g(r)\,\mathrm{d}E\,\mathrm{d}r = \frac{N[E(P,r)]}{\mathcal{N}}\,f(r_\nu)]^{-1}\,\frac{\nu r^{\nu-1}}{b^\nu - a^\nu}\,\mathrm{d}P\,\mathrm{d}r\,. \tag{13}$$

The expression of $p(P)\,\mathrm{d}P$ will be obtained by integrating over r the Eq. (13). The integration limits are easily determined from Fig. 1. Taking into account the two intervals

$$0 \leqslant P \leqslant k_\nu E_1 \tag{14-a}$$

and

$$k_\nu E_1 \leqslant P \leqslant k_\nu E_2\,, \tag{14-b}$$

one has

$$r_{\text{low}} = f_\nu^{-1}\left(\frac{P}{E_2}\right), \qquad r_{\text{upper}} = f_\nu^{-1}\left(\frac{P}{E_1}\right) \tag{15-a}$$

and respectively

$$r_{\text{low}} = f_\nu^{-1}\left(\frac{P}{E_2}\right), \qquad r_{\text{upper}} = b\,. \tag{15-b}$$

Therefore it is

$$p(P)\,\mathrm{d}P = \left\{\int_{f_\nu^{-1}(P/E_2)}^{f_\nu^{-1}(P/E_1)} \frac{\nu r^{\nu-1}}{b^\nu - a^\nu}\,\frac{N[E(P,r)]}{\mathcal{N}}\,f_\nu(r)]^{-1}\,\mathrm{d}r\right\}\mathrm{d}P \qquad \text{for } (0 \leqslant P < k_\nu E_1)\,, \tag{16-a}$$

and

$$p(P)\,\mathrm{d}P = \left\{\int_{f_\nu^{-1}(P/E_2)}^{b} \frac{\nu r^{\nu-1}}{b^\nu - a^\nu}\,\frac{N[E(P,r)]}{\mathcal{N}}\,[f_\nu(r)]^{-1}\,\mathrm{d}r\right\}\mathrm{d}P \quad \text{for } (k_\nu E_1 \leqslant P \leqslant k_\nu E_2)\,. \tag{16-b}$$

Let us determine, for example, the function $p(P)$, defined in Eqs. (16-*a*) and (16-*b*), for the three types of ionization chambers considering a rectangular energy distribution of the ionization events. All energies over the range $0 \leqslant E \leqslant E_0$ are equally probable, so that

$$\frac{N(E)}{\mathcal{N}} = \frac{N_0}{\int_0^{E_0} N_0 \, \mathrm{d}E} = \frac{1}{E_0} . \tag{17}$$

Since in this case $E_1 = 0$ and $E_2 = E_0$, only the interval (14-*b*) has to be taken into account. For the three types of chambers Eq. (8) reads respectively:

plane chamber:

$$r = \frac{P}{k_1 E}(b - a) + a , \tag{18-a}$$

cylindrical chamber:

$$r = a\left(\frac{b}{a}\right)^{P/k_2 E} , \tag{18-b}$$

spherical chamber:

$$r = \frac{ab}{b - (P/k_3 E)(b - a)} , \tag{18-c}$$

and hence the integration limits (15-*b*) are:

plane chamber:

$$r_{\text{low}} = \frac{P}{k_1 E_0}(b - a) + a \qquad r_{\text{upper}} = b , \tag{19-a}$$

cylindrical chamber:

$$r_{\text{low}} = a\left(\frac{b}{a}\right)^{P/k_2 E_0} , \qquad r_{\text{upper}} = b , \tag{19-b}$$

spherical chamber:

$$r_{\text{low}} = \frac{ab}{b - (P/k_3 E_0)(b - a)} , \qquad r_{\text{upper}} = b . \tag{19-c}$$

Finally, the expressions of the function $p(P)$ (16-*b*) come out to be:

plane chamber:

$$p(P) = \frac{1}{k_1 E_0} \int_{P/k_1 E_0 (b-a) + a}^{b} \frac{\mathrm{d}r}{r - a} , \tag{20-a}$$

cylindrical chamber:

$$p(P) = \frac{2}{k_2 E_0} \frac{\ln b/a}{b^2 - a^2} \int_{a(b/a)^{P/k_2 E_0}}^{b} \frac{r}{\ln r/a}\,\mathrm{d}r\,, \tag{20-b}$$

spherical chamber:

$$p(P) = \frac{3}{k_3 E_0} \frac{(1/a) - (1/b)}{b^3 - a^3} \int_{\overline{b - (P/k_3 E_0)(b-a)}}^{b} \frac{r^2}{(1/a) - (1/r)}\,\mathrm{d}r\,. \tag{20-c}$$

Eqs. (20) are plotted in Fig. 2 for different values of b/a. The results expressed in Eqs. (20-*a*) and (20-*b*) have been obtained by WILKINSON [1] following a different method.

We give now a second example for a cylindrical chamber considering a power energy spectrum of the ionization events. These are supposed to have an energy distributed according to the power law

$$N(E)\,\mathrm{d}E = \frac{\text{const}}{E^{\gamma}}\,\mathrm{d}E \quad (\gamma > 1), \tag{21}$$

over the range $E_1 \leqslant E \leqslant E_2$. In this case one has

$$\frac{N(E)}{\mathcal{N}} = (\gamma - 1)\,\frac{1}{E_1^{-\gamma+1} - E_2^{-\gamma+1}}\,E^{-\gamma}. \tag{22}$$

The expression of $p(P)$ obtained by a straightforward calculation reads

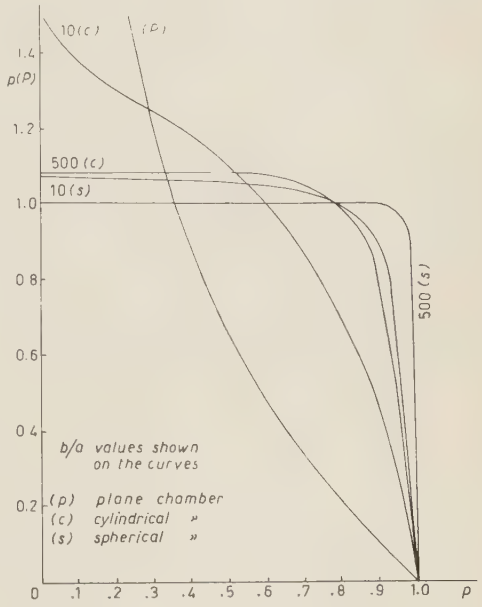

Fig. 2. – Pulse height distributions corresponding to a rectangular energy distribution.

$$\left\{\begin{aligned} p(P) &= \int_{a(b/a)^{P/k_2 E_2}}^{a(b/a)^{P/k_2 E_1}} \frac{2r}{b^2 - a^2}\,(\gamma - 1)\,\frac{1}{E_1^{-\gamma+1} - E_2^{-\gamma+1}}\,P^{-\gamma}\left(\frac{1}{k_2}\,\frac{\ln b/a}{\ln r/a}\right)^{-\gamma+1}\mathrm{d}r \\ &\qquad (0 \leqslant P \leqslant k_2 E_1), \\ p(P) &= \int_{a(b/a)^{P/k_2 E_2}}^{b} \frac{2r}{b^2 - a^2}\,(\gamma - 1)\,\frac{1}{E_1^{-\gamma+1} - E_2^{-\gamma+1}}\,P^{-\gamma}\left(\frac{1}{k_2}\,\frac{\ln b/a}{\ln r/a}\right)^{-\gamma+1}\mathrm{d}r \\ &\qquad (k_2 E_1 \leqslant P \leqslant k_2 R_2). \end{aligned}\right. \tag{23}$$

[1] D. H. WILKINSON: *Ionization chambers and counters* (Cambridge, 1950).

A plot of Eq. (23) is given in Fig. 3 for the particular case of an ionization chamber 60 l in volume and filled with argon at 10 atm, having a $b/a = 100$.

From the general expression of $p(P)$, given in Eqs. (16-a) and (16-b), one may recognize that the dependence on P comes out from the function $N[E(P, r)]$ and also from the integration limits when $E_1 \neq 0$ and $E_2 \neq \infty$. It has to be remarked that only when $E_1 = 0$ and $E_2 = \infty$ the functions $p(P)$ and $N(E)$ have the same shape, because of the proportionality of P to E. In all the other cases distortions occur as a consequence of the presence of P in the integration limits. In this respect the use of cylindrical and spherical chambers is more profitable than the use of parallel plate chambers. Furthermore, in the former ones the higher the ratio b/a is made the more reduced are the distortions.

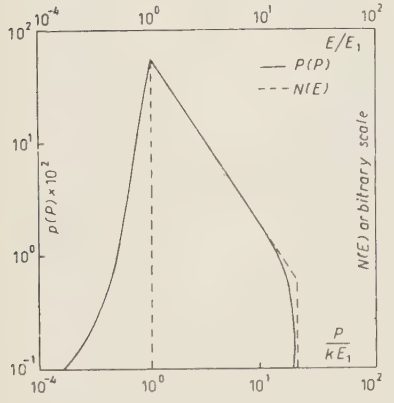

Fig. 3. – Pulse height distribution corresponding to a power law energy distribution for a cylindrical chamber.

From the example of a power energy spectrum, it can be inferred that it is possible to determine the exponent of the power law distribution by extrapolating the slope of the curve $p(P)$ near its maximum.

The determination of $B(P)$, the pulse integral spectrum or « bias curve », follows immediately from the definition itself, that is,

$$B(P) = \int_0^P p(P)\,\mathrm{d}P\,.$$

The same conclusions valid for the reproduction of the energy spectrum $N(E)$, may be drawn also for the reproduction of the integral spectrum

$$B(E) = \int_0^E N(E)\,\mathrm{d}E\,.$$

* * *

The writers wish to thank Prof. G. Poiani for stimulating discussions on the subject of this note.

One of the authors (G.P.) thanks the University of Trieste for financial support and the Director of the Institute of Physics, Prof. P. Budini, for hospitality and for having extended to him staff facilities.

RIASSUNTO (*)

Si è eseguito un esame analitico dell'influenza dei fattori geometrici di una camera di ionizzazione a impulsi sulla corrispondenza tra gli spettri dell'energia e quelli degli impulsi. Si danno formule generali per camere piane, cilindriche e sferiche. Il metodo si presta ad essere usato in pratica e se ne danno esempi d'applicazione.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Measurement of the Absorption in Graphite of Electrons from Muon Decays.

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(ricevuto il 25 Novembre 1957)

Survey articles by KATZ and PENFOLD [1] and PRICE [2] dealing with the measurement of the ionization losses of charged particles show that there are not much experimental data for the energies $E/m_0c^2 > 40$ and that the divergence of the results makes it difficult to establish which variant of the theory is in agreement with experiment.

In the present work measurements were made of absorption in graphite of electrons from muon decays and the result is compared with the calculated absorption curves. The experimental arrangement consisted of a three-tray vertical telescope giving off a stream of muons, a graphite moderator of size 5 cm × 20 cm × 40 cm placed beneath the telescope, and four side trays recording electrons. The triple-coincidence pulse from the telescope was fed with a 1 μs delay (to eliminate knock-on electrons) to a second coincidence circuit where it was shaped into a rectangular pulse 5 μs wide. A coincidence of this pulse with pulses from the two side trays was interpreted as a decay of a muon with the emission of an electron. In order to obtain information on the operation of the arrangement, measurements were made of the mean life of muons in graphite. The value obtained was $2.15 \cdot 10^{-6}$ μs, in agreement with the published results. The stability of operation of the arrangement was checked by means of auxiliary statistics (coincidences of pulses of the telescope and side trays without delay and the third tray of the telescope with the side trays). Measurements of the background were

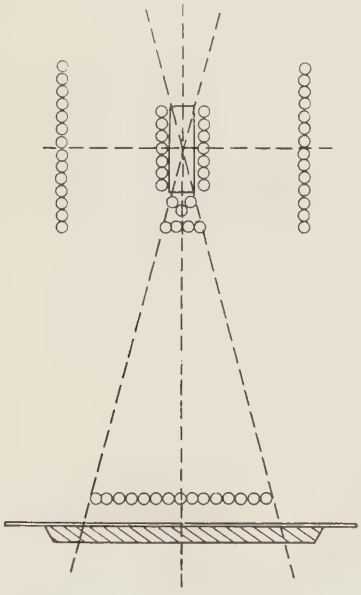

Fig. 1.

[1] L. KATZ and S. PENFOLD: *Rev. Mod. Phys.*, **24**, 28 (1952).
[2] B. T. PRICE: *Rep. on Progress in Physics* (1956).

made six times in the course of 400.25 h. They gave a value of 0.027 coincidences per hour and correspond to the predicted number of chance coincidences. The experimental curve was obtained on the basis of measurements made during 1398 h (749 coincidences).

The expected absorption curve was calculated under the assumption that a linear relation exists between the actual electron path and the energy $Z = \alpha E$ on the basis of the Michel ([3]) spectrum ($\varrho = 0.55$). It was assumed that the actual path travelled by an electron emitted from a given element of moderator volume and reaching the outer side tray is the mean path in graphite $\bar{S}$ (averaged to take into account the angles of emission) increased by some value as a result of scattering. The amount of increase was determined by an approximation method generalizing for thick absorbers the formula of Yang ([4]); a formula was obtained expressing the ratio of the length of path of an electron T to the maximum thickness of absorber t which it can traverse as a function of the initial energy E

$$\frac{T}{t} = \frac{E + 1.4}{E} .$$

Under the above assumptions the number of electrons which trigger the side trays is expressed by

$$N(D,\alpha) = \int \Delta n^* \Theta \left\{ 1 - \frac{k_1}{\alpha^3}\left[\frac{\bar{S}}{2}\left(1 + \sqrt{1 + \frac{5.6\alpha}{\bar{S}}}\right)\right]^3 + \frac{k_2}{\alpha^4}\left[\frac{\bar{S}}{2}\left(1 + \sqrt{1 + \frac{5.6\alpha}{\bar{S}}}\right)\right]^4 \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z ,$$

where D is the absorber thickness, $\bar{S} = \bar{S}(\Theta)$, Δn^* is the number of electrons emitted from a given element of moderator volume, Θ is the solid angle subtended by a given element of the side trays, $k_1 = (4 - \frac{8}{3}\varrho)\cdot W^{-3}$, $k_2 = (\frac{8}{3}\varrho - 3)\cdot W^{-4}$ (W is the maximum energy in the spectrum). The integration is performed over the moderator volume occupied by the stream of muons. The calculation of the above integral was carried out numerically. A correction was made for the points ($\alpha = 0.50, D = 0$, 6, 8, 10 cm and $\alpha = 0.46$, $D = 0$, 8, 10 cm) determined in this way. This correction results from taking the mean path for all electrons emitted from a given element.

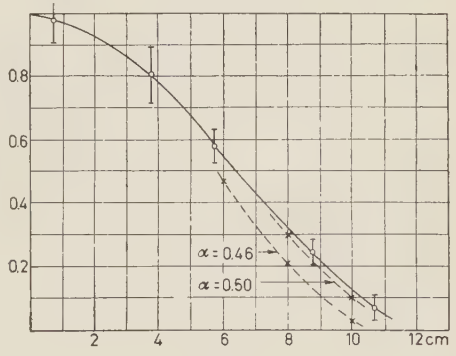

Fig. 2.

On the basis of the comparison of the experimental curve with the calculated curves it was established that the best agreement is obtained for $\alpha = 0.52$. This value is the inverse of the mean total losses of the electron over an actual path $1/\alpha = \mathrm{d}E/\mathrm{d}Z = 1.92\ \mathrm{MeV/g\,cm^{-2}}$. To compare the experimental result with the theoretical results for ionization loss, the contribution of the radiation loss should

([3]) L. Michel: *Proc. Phys. Soc.*, A **63**, 514 (1951).
([4]) C. N. Yang: *Phys. Rev.*, **84**, 599 (1951).

be estimated. Since with a small error we can write for the investigated energy interval.

$$\left(\frac{\mathrm{d}E}{\mathrm{d}Z}\right)_{\mathrm{ion}} = \mathrm{const} \qquad \text{and} \qquad \left(\frac{\mathrm{d}E}{\mathrm{d}Z}\right)_{\mathrm{rad}} \approx E, \qquad \text{then} \qquad -\left(\frac{\mathrm{d}E}{\mathrm{d}Z}\right) = A + B \cdot E\,,$$

therefore

$$Z = \frac{1}{B} \ln\left(1 + \frac{B}{A} \cdot E\right).$$

Using this formula instead of $Z = \alpha \cdot E$ calculations were made for an absorption curve taking into account three elements of volume (lying on an axis perpendicular to the side trays) for a fixed parameter $B = 1.7 \cdot 10^{-2}\,\mathrm{cm^2/g}$ for $A = 1.54$, 1.67, and 1.82. The best agreement between the experimental results and the calculated curve is obtained for $A = 1.70\,\mathrm{MeV/g\,cm^{-2}}$. Since the mean radiation loss constitutes 13% of the ionization loss the possibility of the production of secondary electrons by Bremsstrahlung quanta was therefore estimated. This effect was of no essential importance. The result $(\mathrm{d}E/\mathrm{d}Z) = (1.70 \pm 0.15)\,\mathrm{MeV/g\,cm^{-2}}$ is in agreement within the limits of error with the predicted losses computed according to the Halpern-Hall formula for the mean energy in the spectrum of electrons from muon decays.

It seems that the differences between the theoretical and experimental results discussed in the KATZ and PENFOLD article are due mainly to the neglect, for low energies, of the increase in path caused by scattering, and, for relativistic energies, of the radiation loss.

About a Certain Method for Finding the Asymptotic Phases.

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(ricevuto il 15 Gennaio 1958)

The radial wave equation of the scattering problem, when a Coulomb-potential is included in the interaction we can write in the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\mathcal{R}_l}{\mathrm{d}r}\right)+\left[k^2-\frac{C}{r}-\frac{l(l+1)}{r^2}-U(r)\right]\mathcal{R}_l=0\,. \tag{1}$$

We assume that $U(r)$ vanishes more rapidly than $1/r$ as r tends to infinity and for $r=0$, $U(r)$ is not more singular than $1/r^p$, where $1\leqslant p<2$. For $U=0$ the Schrödinger equation (1) has following independent solutions ${}_1\mathcal{R}_l$ and ${}_2\mathcal{R}_l$ which when r tends to infinity behave as given below

$$\begin{cases} {}_1\mathcal{R}_l \to (kr)^{-1}\sin\left(kr-\dfrac{l\pi}{2}+\eta_{c,l}-\alpha\ln 2kr\right) \\ {}_2\mathcal{R}_l \to (kr)^{-1}\cos\left(kr-\dfrac{l\pi}{2}+\eta_{c,l}-\alpha\ln 2kr\right), \end{cases} \tag{2}$$

where

$$\eta_{c,l}=\arg\Gamma(i+l+i\alpha) \qquad \text{and} \qquad C=2\alpha k\,. \tag{3}$$

When $U(r)\not\equiv 0$ the solution $\mathcal{R}_l$ of eq. (1) has following asymptotic behaviour

$$\mathcal{R}_l \to (kr)^{-1}\sin\left(kr-\frac{l\pi}{2}+\eta_{c,l}+\eta_{u,l}-\alpha\ln 2kr\right), \tag{4}$$

where $\eta_{u,l}$ is the asymptotic phase caused by the interaction $U(r)$.

As it is shown by Jost, Wheeler and Breit [1] the independent solutions ${}_1\mathcal{R}_l$ and ${}_2\mathcal{R}_l$ of the Schrödinger eq. (1) for $U(r)\equiv 0$ which have the asymptotic

[1] N. F. Mott and H. S. W. Massey *The Theory of Atomic Collision*, (Oxford, 1949), 2nd edition Chap. III.

behaviour as given by eq. (2) are

$$(5)\quad \begin{cases} {}_1\mathcal{R}_l = \exp\left[-\frac{1}{2}\pi\alpha\right] \cdot \frac{\Gamma(l+1+i\alpha)}{(2l+1)!} \cdot \\ \qquad \cdot (2kr)^l \exp[ikr]\, {}_1F_1(i\alpha + l + 1,\ 2l+2,\ -2ikr)\,, \\ {}_2\mathcal{R}_l = \frac{1}{\pi}\,(\exp[2\pi\alpha] - 1)\cdot \\ \qquad \cdot \left\{\ln 2kr + 2\gamma - \sum_{s=1}^{U+1} s^{-1} + \sum_{s=1}^{l} \frac{s}{s^2+\alpha^2} + \text{P.R.}\frac{\Gamma'(i\alpha)}{\Gamma(i\alpha)}\right\} {}_1\mathcal{R}_l + \mathcal{H}_l\,, \end{cases}$$

where

$$\mathcal{H}_l = 2\exp\left[\frac{1}{2}\pi\alpha\right] \text{P.R.}\exp[i(kr - \eta_{l,c})]\left\{\sum_{s=0}^{2l} \frac{(-i)^s(2l-s)!}{s!\,\Gamma(k-s+1-i\alpha)}\,(2kr)^{s-l-1} - \right.$$

$$\left. -\frac{1}{\pi}\sinh\pi\alpha \sum_{s=1}^{\infty}(-i)^s \frac{\Gamma(l+1+s+i\alpha)}{(2l+1+s)!\,s!}\,(2kr)^{s+l}\,a_{l,s}\right\}$$

and

$$(6)\qquad a_{ls} = \sum_{t=1}^{s}\left(\frac{1}{t} + \frac{1}{2l+1+t} - \frac{1}{l+t+i\alpha}\right).$$

In equation (5) and (6) P.R. denotes « real part of », γ is Euler's constant and the symbol ${}_1F_1$ denotes the confluent hypergeometric function. ${}_1\mathcal{R}_l$ is the bounded solution for $r=0$ and ${}_2\mathcal{R}_l$ is the unbounded solution for $r=0$.

Up to that point $r=r_1$, where $U(r)$ begins to be negligible in comparison to C/r, the solution $\mathcal{R}_l$ of the Schrödinger eq. (1) may be written as

$$(7)\qquad \mathcal{R}_l(r) = C_1\,{}_1\mathcal{R}_l + C_2\,{}_2\mathcal{R}_l\,,$$

where C_1, C_2 are constants.

Taking into consideration equations (2) and (4) we can write $\mathcal{R}_l$ given by eq. (7) for $r > r_1$ as follows

$$(8)\qquad \mathcal{R}_l(r) = \text{const}\,[\cos\eta_{u,l}\,{}_1\mathcal{R}_l(r) + \sin\eta_{u,l}\,{}_2\mathcal{R}_l(r)]\,.$$

Since $\mathcal{R}_l(r)$ has a infinite number of zero's then if we denote by r_0, such zero of $\mathcal{R}_l$ for which is fullfilled that $r_0 > r_1$ then we find

$$(9)\qquad \text{tg}\,\eta_{u,l} = -\frac{{}_1\mathcal{R}_l(r_0)}{{}_2\mathcal{R}_l(r_0)}\,.$$

The last formula gives us the asymptotic phase $\eta_{u,l}$ if the zero r_0 of $\mathcal{R}_l(r)$ is known. We can find numerically the zero $r_0 > r_1$, e.g. by the simple method proposed by Löwdin and Sjölander [2]. The method given here is very accurate when one will obtain the accurate values of the asymptotic phases $\eta_{u,l}$. The Schrödinger equation (1) has an important role in several problems [3].

[2] P. O. Lowdin and A. Sjolander: *Ark. f. Fys.*, **3**, 155 (1951).
[3] S. M. Shah: *Proc. of the Phys. Soc.*, **68**, 947 (1955).

Some Experimental Results on the Angular Correlations in the π-μ-e Decay.

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Institute for Theoretical Physics, University of Copenhagen, Denmark

(ricevuto il 23 Gennaio 1958)

It is now well known that in the successive decays

$$\pi^+ \to \mu^+ + \nu$$

$$\mu^+ \to e^+ + \nu + \bar{\nu}$$

the positron is emitted in a direction which is distributed asymmetrically with respect to the direction in which the μ-meson is emitted. It is also well known that the distribution function for the angle ϑ between the two directions is of the form

$$(1 + A \cos \vartheta)\, \mathrm{d}(\cos \vartheta) .$$

The coefficient A has been measured in many laboratories; however, the agreement between the results is not quite satisfactory. In D. H. Wilkinson's analysis [1] of the results obtained by eight groups of workers (including part of our results) a χ^2 test on the values of A, ranging from -0.03 ± 0.04 to $-0.22 \pm \pm 0.07$, gives a Pearson probability of $P = 0.10$.

Similar problems of poor consistency have arisen in the course of the work performed in this Institute, and will be described in the following.

π-μ-e decays were studied in a stack of photographic emulsions which had been exposed to cosmic radiation about 30 km over South England. The dimensions of the stack were $20 \times 15 \times \times 4.8\ \mathrm{cm}^3$. Part of this stack had earlier been scanned for mesonic events. In the scan-sheets we had an easily available number of π-μ-e decays. 796 decays fulfilled the requirement that the μ-meson stopped more than 15 μm (shrunk emulsion) from the surfaces of the emulsion. In 29 of these decays no positron track was seen. In the remaining 767 decays, the angle, ϑ, between the μ-meson track at the π-μ decay point and the positron track at the μ-e decay point was measured carefully. The distribution in ϑ is shown in Fig. 1 (dotted line). From this distribution the value $A = -0.25 \pm 0.08$ [2] is obtained.

To check the data for systematic errors, the distributions of dip angles for the μ-mesons and the positrons were examined and nothing unexpected was

(1) D. H. Wilkinson: *Nuovo Cimento*, **6**, 516 (1957).

(2) Reported in a preliminary stage at the Bristol meeting (Feb. 1957).

observed. The possibility of an observational correlation between ϑ and the steepness or depth of the tracks was also examined, but no indication of such an effect could be detected.

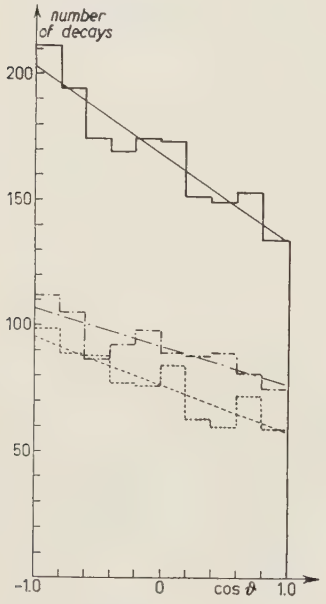

Fig. 1. – ϑ-distributions. ... first experiment; 767 decays determining $A = -0.25 \pm 0.08$; - - - second experiment; 915 decays determining $A = -0.17 \pm 0.07$; — combined distributions; 1682 decays determining $A = 0.21 \pm 0.05$. For each distribution a line corresponding to the fitted distribution function $1 + A \cos \vartheta$ is drawn.

However, when the material was divided into seven groups corresponding to the work of the individual observers, the consistency between the corresponding values of A was not satisfactory. A χ^2 test gave a probability of $P = 0.03$. This is difficult to understand, unless the observers, even with the low magnification (10×10) used in the scan, were influenced by the relative configuration of the μ- and e- tracks in recording π-μ-decays. In this connection we wish to emphasize that the scans were performed prior to any discussion of the ϑ-distribution; therefore a possible bias cannot be due to definite expectations regarding the asymmetry of this distribution.

In order to avoid any subjective bias, a new experiment was performed special care being taken during the scanning stage. Other plates from the same stack were scanned for stopping mesons with a grey secondary. The only criterion used for the selection was a clear, true change in the density of a track of a slow meson. Obviously, with this method, besides π-μ-decays two types of events were recorded, viz. one-prong σ-stars and inelastically scattered mesons, but neither of these types could be mistaken for a π-μ-decay in the subsequent measuring stage. In this way, 919 events, in which the μ-meson stopped more than 15 μm from the surfaces of the emulsion, were obtained. In about 400 of them, the positron track was identified by two independent observers; no mistake was observed. In four cases, it has been impossible to find the positron track. Fig. 1 shows (dashed dotted line) the distribution of ϑ giving $A = -0.17 \pm 0.07$ ([3]).

Also here the values of A based on the work of the different observers exhibit a poor consistency. A χ^2 test gave $P = 0.05$. Therefore, we do not think that we are justified in disregarding the first experiment and accepting the second one; combining both experiments we get a ϑ-distribution as shown by the full line in Fig. 1. The resulting value of A is then $A = -0.21 \pm 0.05$.

The fact that the poor consistency persisted in spite of the efforts to eliminate bias, together with the large fluctuations in the measurements from different laboratories, suggest that the lack of consistency might have its origin in the presence of some physical parameter which has not been taken into account. A possible, though not very likely, explanation would be the existence of a

([3]) Reported in M. F. KAPLON's review at the *Rochester Conference* (1957) and in ([1]).

correlation between A and the angle formed by the μ-meson momentum with

A least square fit of a straight line $A=\langle A\rangle+\beta\cos\psi$ to the points gives

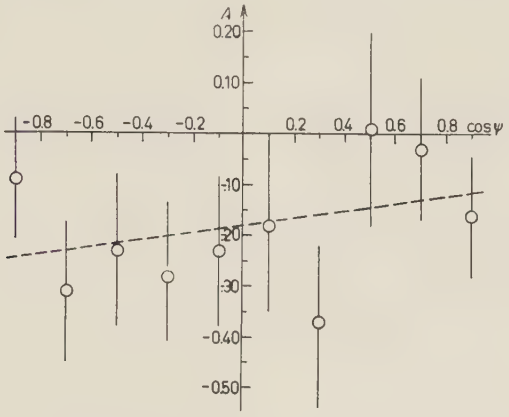

Fig. 2. – The values of A measured at different angles, ψ, between the direction of emission of the μ-meson and the vertical.

that of the particles producing the π-mesons.

We therefore attempted to find a correlation between ϑ and the angle ψ between the μ-meson momentum and the vertical, which was taken as representing roughly the direction of the incoming cosmic ray primaries.

The result from about 1500 decays appears in Fig. 2, where the measured A's are presented as a function of $\cos\psi$. $\beta=0.07\pm0.08$. However, as is immediately obvious from the diagram, the points are consistent with $\beta=0$.

Thus, even if the present material does not allow us to state whether there is an effect of this type, we may conclude that the effect is not large enough to explain the lack of internal consistency of our material, nor to account for the bad agreement between the results of different laboratories.

Double Scattering of Electrons.

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National Physical Research Laboratory
of the Council for Scientific and Industrial Research - Pretoria, South Africa

(ricevuto il 22 Marzo 1958)

Although many experiments have been described on this subject, very few reliable results are known until now ([1-3]). Whereas the older publications mainly deal with the problem of finding the effect of azimuthal asymmetry, the various experimental difficulties gradually became apparent. A good compilation and discussion of these difficulties are given by C. G. Shull, C. T. Chase and F. E. Myers ([2]), as well as a value for the asymmetry at 400 keV electron energy. Norio Ryo ([1]) presents asymmetry measurements at lower energies at angles between $\theta = 105°$ and $\theta = 135°$ where the maximum effect is expected. However, these values seem to be only half the asymmetry predicted by theory ([1,4]). It therefore seemed worthwhile to improve the experiment if possible, and to extend the measurements to $\theta = 90°$.

To overcome the difficulties of multiple scattering ([5]), we used only one counter at the transmission side of the second scattering foil; to monitor this detector, we introduced a second counter which measured the scattered particles coming from the reflection side of the first foil. As Fig. 1 shows, this monitor consists again of a scattering assembly.

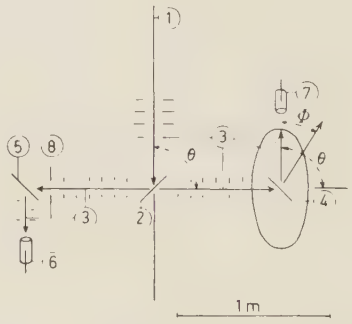

Fig. 1. – Arrangement of foils and detectors. 1) Beam from electron gun. 2) First scattering foil. 3) Collimators. 4) Analysing foil. 5) Scattering foil. 6) Monitor, G.M. counter in fixed position. 7) Analysing G.M. counter defining the azimuthal angle Φ. 8) Variable aperture.

([1]) N. Ryo: *Journ. Phys. Soc. Japan*, **7**, 125, 130 (1952).

([2]) C. G. Shull, C. T. Chase and F. E. Myers: *Phys. Rev.*, **63**, 29 (1943).

([3]) W. H. Louisell, R. W. Pidd and H. R. Crane: *Phys. Rev.*, **94**, 7 (1954).

([4]) C. B. O. Mohr and L. J. Tassie: *Proc. Phys. Soc.*, A **67**, 711 (1954).

([5]) G. Goertzel and R. T. Cox: *Phys. Rev.*, **63**, 37 (1943).

In fact, the monitoring part is made precisely symmetrical with respect to the analysing system except that its position remains fixed, and a variable aperture is placed between the collimator and the foil. With this aperture, the counting rate of the monitor is adjusted to give the same counting rate as the analysing detector at $\Phi = 90°$ or $270°$. ($\Phi = 0$, anti-parallel with respect to the beam coming from the electron gun). At Φ values, other than 90° and 270°, a correction has been made for the counting loss due to the dead time of the counters. It can easily be shown how this affects the result, and the counting rates were so chosen that this correction factor, to be applied on the error. Special attention has been given to the fact that electron guns often tend to emit particles not statistically distributed on the time axis, but in clusters following each other with the frequency of the mains. This strongly affects the counting loss and therefore the result. A special electron gun was therefore constructed. Test runs were made to control this effect, and also during all measurements, the distribution of the counts were observed on the screen; for each point 10^5 counts were taken.

With this technique, the asymmetry of gold-gold scattering was measured and the result is shown in Fig. 2. The gold foils were $1.1 \cdot 10^{-5}$ cm thick; the effective thickness is therefore $1.5 \cdot 10^{-5}$ cm,

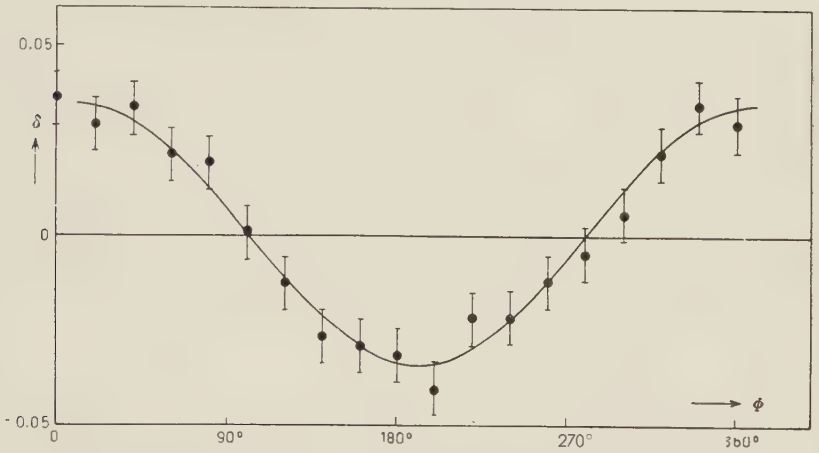

Fig. 2. – Azimuthal asymmetry obtained for gold-gold double scattering at 80 keV. The solid line represents the best fitting cosine curve obtained by harmonic analysis.

asymmetry, is of the order of 1.06. An ordinary X-ray high voltage supply was used. Stability of energy was achieved by stabilising the primary voltage. Furthermore, continuous recording of the variations of the high voltage guaranteed full control. After a ten hour run, the counting rate remained constant to within 1%. Alignment was checked with test runs using hydrocarbon foils, and the asymmetry was found to be within the statistical fulfilling the Wentzel-criterion [6]. As detectors for these experiments, Halogen counters (N.R.D. type DR3) were used.

This curve (Fig. 2) shows the expected cosine dependence. Harmonic analysis [7] gives a phase shift of $\Phi + 11°$. This phase shift is much too big to be

[6] G. WENTZEL: *Ann. d. Phys.*, **69**, 335 (1922)

[7] S. CHAPMAN and J. BARTELS: *Geomagnetism* (Oxford, 1940).

ascribed to misalignment of the apparatus. On the other hand, in view of the relatively large errors, the effect is not marked enough to be regarded as real. If we define δ by

$$I = I_0(1 + \delta \cos \Phi) ,$$

where I = the measured intensity and I_0 = a constant, harmonic analysis of

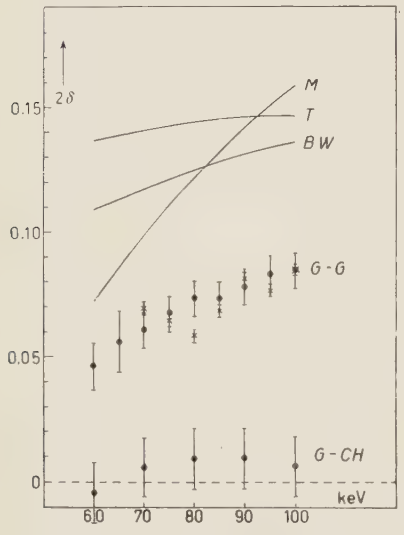

Fig. 3. – Azimuthal asymmetry as a function of energy. M is the theoretical curve from Mott ([8]), BW that from Bartlett and Watson ([9]) and T that of Tassie ([10]). The formula of Tassie has been used without screening correction. G-G represents our results obtained for gold-gold scattering. The crosses (x) indicate results obtained from harmonic analysis. G-CH denote values obtained from gold-hydrocarbon foil measurements.

([8]) N. F. MOTT: *Proc. Roy. Soc.*, **124**, 425 (1929); **135**, 429 (1932).

([9]) J. H. BARTLETT and R. E. WATSON: *Phys. Rev.*, **56**, 612 (1939).

([10]) L. J. TASSIE: *Phys. Rev.*, **107**, 1452 (1957).

the uncorrected gold-gold scattering results at 80 keV gives

$$2\delta = 0.068 \pm 0.0024 .$$

Similar runs have been made at different energies.

It is obvious that the harmonic analysis reduces the statistical error, so that it becomes small compared to the remaining errors due to misalignment etc. It is therefore misleading to give the energy dependence with errors obtained from such an analysis, as long as there is no better knowledge about the remaining systematic errors. In Fig. 3, the 2δ values are given for different energies; these were obtained only from the counting rate at $\Phi = 0°$ and $\Phi = 180°$. Here the errors are purely statistical. Theoretical curves, shown for comparison, indicate that our measurements give also only about half the theoretically predicted asymmetry. Unfortunately there are no other experimental values available in this energy region. However, our results are in good agreement with those obtained by an extrapolation to $\theta = 90°$ of the experimental values given by Ryo. Further experiments at higher energies are in progress.

* * *

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Asymptotical Behaviour of Higher Green Functions.

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(ricevuto il 5 Aprile 1958)

The asymptotical behaviour of higher Green functions in the ultraviolet region has been recently studied by Konuma and Umezawa [1] by means of renormalization invariance considerations. Those authors used the concepts of « renormalized fields » and « cut-off momenta » that complicated their analysis and led to some errors.

A simpler mode of investigation is the renormalization group technique [2] which reduces the problem to the solution of Lie differential equations. We have carried out such an investigation and have obtained the following results (a detailed account of this investigation will be published in the new journal *Dokl. Vysshey Shkoly SSSR*).

In some important cases the asymptotical behaviour of higher Green functions is actually described by formulae of type (19), (20) from paper I. But generally it is necessary to take into account the fact that higher Green functions depend on many scalar momentum arguments and, correspondingly, obey various forms of asymptotics. We have obtained a general recipe for determining these different asymptotics. This recipe reduces to perturbation theory calculation to one order higher than indicated in I and to subsequent solution of the Lie differential equations. It turned out that Konuma and Umezawa's rules for defining the coefficients S in their formulae (19), (20) are correct only for the symmetrical momentum asymptotics (i.e. for the case when all scalar arguments simultaneously tend to the same large limiting value). In more general cases these coefficients are to be determined from perturbation theory calculations of the next order.

As a simple illustration let us consider the 4-boson Green function $\square$ in the theory

$$L(x) = \frac{h}{4}\varphi^4(x)\,,$$

which is defined by

$$\frac{1}{S_0}\left\langle \frac{\delta^4 S}{\delta\varphi(p')\,\delta\varphi(q')\,\delta\varphi(-p)\,\delta\varphi(-q)} \right\rangle = \delta(p'+q'-p-q)\,\frac{6ih}{(2\pi)^4}\,\square(p', q', p, q; h)\,.$$

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[1] M. Konuma and H. Umezawa: *Nuovo Cimento*, **4**, 1461 (1956), referred to as I.

[2] See for instance N. N. Bogolioubov and D. V. Shirkov: *Introduction to quantized field theory* (Moscow, 1957); I. F. Ginzburg: *Dokl. Akad. Nauk SSSR*, **110**, 535 (1956), in Russian. See also N. N. Bogolioubov and D. V. Shirkov: *Nuovo Cimento*, **3**, 845 (1956); V. Z. Blank and D. V. Shirkov: *Nucl. Phys.*, **2**, 365 (1956-1957) in English.

The second order perturbation calculation leads to

$$\square(p', q', p, q; h) = 1 - \frac{3h}{4\pi^2} \ln \left[\frac{(p'-p)^2(p-q')^2(p+q)^2}{m^6} \right].$$

It is already evident from this expression that $\square$ has various asymptotics.

Setting

$$\square(p', q', p, q; h) = \widetilde{\square}(p'^2, q'^2, p^2, q^2, (p-p')^2, (p+q)^2; h),$$

we can define various asymptotics of $\widetilde{\square}$. For instance

$$\square_1(x; h) = \widetilde{\square}(x, y, y, y, y, y; h),$$

$$\square_2(x; h) = \widetilde{\square}(y, y, y, y, y, x; h),$$

$$\square_3(x; h) = \widetilde{\square}(x, x, x, x, x, x; h),$$

as $x \gg y \sim m^2$.

The integration of the Lie equations yields

$$\square_1(x; h) = \left(1 + \frac{9h}{4\pi^2} \ln x^{-1}\right)^{-\frac{1}{3}}, \qquad \square_2(x; h) = \left(1 + \frac{9h}{4\pi^2} \ln x\right)^{-\frac{2}{3}},$$

$$\square_3(x; h) = \left(1 + \frac{9h}{4\pi^2} \ln x^{-1}\right)^{-1}.$$

Thus, only the symmetrical asymptotics $\square_3$ corresponds to the results of paper I.

It is essential that the physically interesting asymptotics, which corresponds to scattering of real particles (external $p_i^2 = m_i^2$) with high energy, are not symmetrical. So in general they are not described by Konuma and Umezawa's formulas.

Note added in proof (20 May 1958).

Another counter-example to the rules of paper I is given by the two-particle scattering matrix element in Thirring's soluble model ([3]). The renormalization group technique leads to an expression for this matrix element ([4]) which coincides with the correct one for small values of the coupling constant.

([3]) W. E. THIRRING: *Annals of Phys.* **3**, 91 (1958).

([4]) M. E. MAYER and D. V. SHIRKOV: *JINR-Preprint* P-187 (a short account will be published in *Dokl. Akad. Nauk SSSR*).

A Suggestion on the Theory of the $\pi \to e + \nu$ to $\pi \to \mu + \nu$ Ratio.

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(ricevuto il 6 Aprile 1958)

1. – The main difficulty for any of the so far proposed theories of universal Fermi interaction is the explanation of the relative rates for $\pi \to \mu + \nu$ and $\pi \to e + \nu$. These decays are supposed to proceed through virtual baryon pairs and, although their rates cannot be calculated at present, the ratio between the two rates can be uniquely predicted for the different Fermi interactions. For S, V or T interaction both decays are forbidden; for a PS interaction the predicted rates are of the same order of magnitude; for A interaction the ratio of $\pi \to e + \nu$ to $\pi \to \mu + \nu$ is found to be $(m_e/m_\mu)^2 (m_\pi^2 - m_e^2)^2/(m_\pi^2 - m_\mu^2)^2 = 13.6 \cdot 10^{-5}$. Experimentally, Anderson and Lattes find only a 1% probability that the ratio could be larger than $2.1 \cdot 10^{-5}$ [1].

Feynman and Gell-Mann [2], and also Marshak and Sudarshan [3], have suggested a theory of weak interactions which leads to a coupling $A \pm V$. This theory seems capable of explaining in a simple way most of the experimental results on weak interactions. At present, the main difficulty for the theory lies in the discrepancy for the $\pi \to e + \nu$ to $\pi \to \mu + \nu$ ratio, which is obviously predicted to be the same as for pure A coupling. Other possible experimental tests of the hypothesis of a universal Fermi interaction in the $A \pm V$ theory have been proposed [4], but no sufficient data are available at present.

One can think of different modifications of the $A \pm V$ coupling that may be able to remove the discrepancy with $\pi \to e + \nu$ and, at the same time, leave the other successful predictions essentially unchanged. A most obvious possibility is the removal from the theory of the hypothesis of a universal interaction, meant in the sense of strict equality of the coupling constants for various processes. Another possibility is the addition to the universal $A \pm V$ interaction of a small pseudoscalar part which could interfere destructively in

(*) John Simon Guggenheim Foundation Fellow, on leave from the University of California, Berkeley, California.

(1) H. L. Anderson and C. M. G. Lattes: *Nuovo Cimento*, **6**, 1356 (1957).

(2) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).

(3) R. E. Marshak and E. C. G. Sudarshan: *Phys. Rev.* (to be published).

(4) R. Gatto: *Phys. Rev.* (to be published).

$\pi \to e+\nu$ in such a way as to exactly, or almost exactly, cancel the A contribution, while contributing very little to the larger $\pi \to \mu+\nu$ rate.

We want here to point out the possibility of a different approach to the problem which seems to us rather suggestive if one wants to insist on the hypothesis of universal A $\pm$ V coupling.

2. – We first show an equivalence theorem. Let us call $m_e^{(B)}$ and $m_\mu^{(B)}$ the bare masses of the electron and of the muon, that are obtained in the limit when all electromagnetic interactions are switched off. It can then be shown that in the predicted relative rate of $\pi \to e+\nu$ to $\pi \to \mu+\nu$ with an A coupling the mass ratio m_e/m_μ is really the unknown ratio of the bare masses, $m_e^{(B)}/m_\mu^{(B)}$, which would obtain in the limit $e^2 = 0$. The theorem is true for ordinary quantum electrodynamics and also if the photon propagator, K^{-2}, is replaced by other functions $f(K^2)$.

For the derivation we assume that the intermediate baryon states have sufficiently large total energy that the decay interaction can be approximated by a local interaction (the pion disappears at the same space-time point where the lepton pair is created). The matrix element for decay through A coupling is then effectively (l denotes the charged lepton, e or μ)

$$G\langle \nu | \gamma_5 \gamma_\mu \frac{\partial \varphi^*}{\partial x^\mu} | l \rangle \,. \tag{1}$$

If the only electromagnetic interaction of the charged lepton is with some arbitrary *external* electromagnetic field A^{ext}, then the lepton wave function is assumed to satisfy

$$\gamma_\mu (p_\mu - e A_\mu^{\text{ext}}) \psi_l = m_l^{(B)} \psi_l \,, \tag{2}$$

with, of course, the bare mass $m_l^{(B)}$, which the lepton would have if $e^2 = 0$. It follows [5] that

$$\left\langle \nu | \gamma_5 \gamma_\mu \left(\frac{\partial}{\partial x^\mu} + ie A_\mu^{\text{ext}} \right) \varphi^* | l \right\rangle = m_l^{(B)} \langle \nu | \gamma_5 \varphi^* | l \rangle \,, \tag{3}$$

where the boson and the lepton wavefunctions are both those appropriate to the presence of A^{ext}. Since $\langle \nu | \gamma_5 \varphi^* | l \rangle$ is not very sensitive to the mass of l, we see that in the presence of A^{ext} the ratio of the matrix elements for e and for μ decay is proportional to $m_e^{(B)}$ or $m_\mu^{(B)}$ respectively, just as if $A^{\text{ext}} = 0$. This is true in an *arbitrary* external electromagnetic field, and is easily shown to remain valid if, instead of a scattering in the external field at some point, a real or virtual photon is emitted or absorbed at that point. Moreover the virtual photons can be absorbed after propagating according to an arbitrary Green's function. Thus, it is still true that the matrix element (1), when corrected for its coupling to the electromagnetic field is $m_l^{(B)}$ times the matrix element $\langle \nu | \gamma_5 \varphi^* | l \rangle$ to any order in e^2. The latter matrix element is relatively insensitive to the lepton mass and so the ratio of the matrix element for $\pi \to e+\nu$ to that of $\pi \to \mu+\nu$ is essentially independent of whether or not e^2 is zero and depends only upon the ratio of the bare masses. This result is exact if the sources of the electromagnetic field are the conventional currents and if (1) is considered as an exact representation of the interaction, but is independent of other modifications to the behaviour of the electromagnetic field.

3. – It is clear in what way the above result suggests a different approach to the problem of the $\pi \to e+\nu$ to the $\pi \to \mu+\nu$ ratio. It may be that the pre-

[5] See S. A. BLUDMAN and M. A. RUDERMAN: *Phys. Rev.*, **101**, 910 (1956); M. A. RUDERMAN and W. K. R. WATSON: *Bull. Amer. Phys. Soc.*, **1**, 383 (1956).

sent form of quantum electrodynamics is only an approximation to a more complete theory, which is only valid at longer wavelengths, or close to the energy shell ($K^2 \cong 0$). We have then to assume, first of all, that the above theorem derived from the formalism of present quantum electrodynamics (with at most a modification of the photon propagator) will still be approximately valid in the complete theory. We then propose that the ratio $m_e^{(B)}/m_\mu^{(B)}$, between the bare masses, is considerably smaller than the ratio m_e/m_μ between the physical masses, such as to give a result for the $\pi \to e + \nu$ to $\pi \to \mu + \nu$ ratio consistent with the measured upper limit ([6]). The quantitative relation between the bare masses and the physical masses is a problem that we leave unsolved, of course. We do not feel, however, that the mere introduction of the covariant cut-off into present quantum electrodynamics gives a significant clue, essentially for two reasons: first, if one wants to give any meaning at all to the perturbative approach to the problem of the self-mass, one finds it necessary to postulate a cut-off corresponding to an enormous mass, of the order of $e^{137}\, m_e$; second, if one insists on introducing a covariant cut-off without any further modification, then the attractive possibility of having a bare electron mass exactly zero would be lost. Finally we want to emphasize that we are not actually offering a solution for the $\pi \to e + \nu$ problem, but only suggesting a different approach which makes the observed decay ratio not incompatible with a universal Fermi interaction.

([6]) We note that the cancellation of the effect of a change in lepton mass from $m^{(B)}$ to m by the other radiative corrections, which occurs for the pion decay with A interaction, does not hold for nuclear β-decay. In the latter case the mass renormalization can be calculated to be a much larger perturbation than other radiative corrections both for virtual photons with $K \simeq 0$ and with $K^2 \gg 0$. In so far as these radiative corrections do *in part* cancel the mass change, the electron polarization will be greater than v/c, calculated with the observed electron mass.

A Case of Double Pair Production by an Electron.

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(ricevuto il 7 Aprile 1958)

During the scanning of G5 emulsions exposed to cosmic rays at 25 km altitude, we have found an event which may be interpreted as the simultaneous creation of two pairs by an electron. It is to be noted that in two instances [1] the process of double pair production by a photon has been observed in nuclear emulsion, both cases discussed by Heitler [2]; while so far no instance has been reported of an event with an electronic primary.

A branch of an electron pair (a, b) originating at O, after traveling 6.7 mm gives off (see Fig. 1) four other fairly well collimated ($1.6 \cdot 10^{-2}$ rad). According to a previous article [3] the mean gap length was used for the ionization measurements: the error is $0.75/\sqrt{N}$, where N is the number of gaps counted. Distorsion was measured locally, in each plate. Scattering measurements being influenced by distorsion, should be considered as only indicative. The given errors are only statistical; track No. 1 is not measurable. In spite of all this, measurements allow us to identify the primary and the secondary particles as very probably electrons.

The event can be simulated by the following casuals:

a) Coincidence of a normal trident and a background pair in the emulsion. The pair density being 1.7 pairs/mm^3, the probability of such casual coincidence comes out to be $\ll 10^{-7}$.

b) Consecutive creation of two tridents (or of a trident and a pseudo-trident) within a distance Δx so small as to render indistinguishable the two points of origin. The second trident may be generated by the primary or by one of the first two secondaries.

To estimate this last casual we have first tried to determine Δx, by means of an accurate grain-count over three different zones of the event. These were the single track b ($n_1 = (18.5 \pm 1.0)$ grains over 60 μm); the region $O'O''$ immediately after the origin O' of the event; and finally the zone in which the five tracks start being distinguishable ($n_3 = 98.0 \pm 3.0$). Results are shown in Fig. 2. While, as is natural, $n_3/n_1 =$

(1) J. E. Hooper and D. T. King: *Phil. Mag.*, **41**, 1194 (1950).

(2) W. Heitler: *Quantum theory of radiation*, 3rd ed. (1954), p. 228.

(3) C. Castagnoli, G. Cortini and A. Manfredini: *Nuovo Cimento*, **2**, 301 (1955).

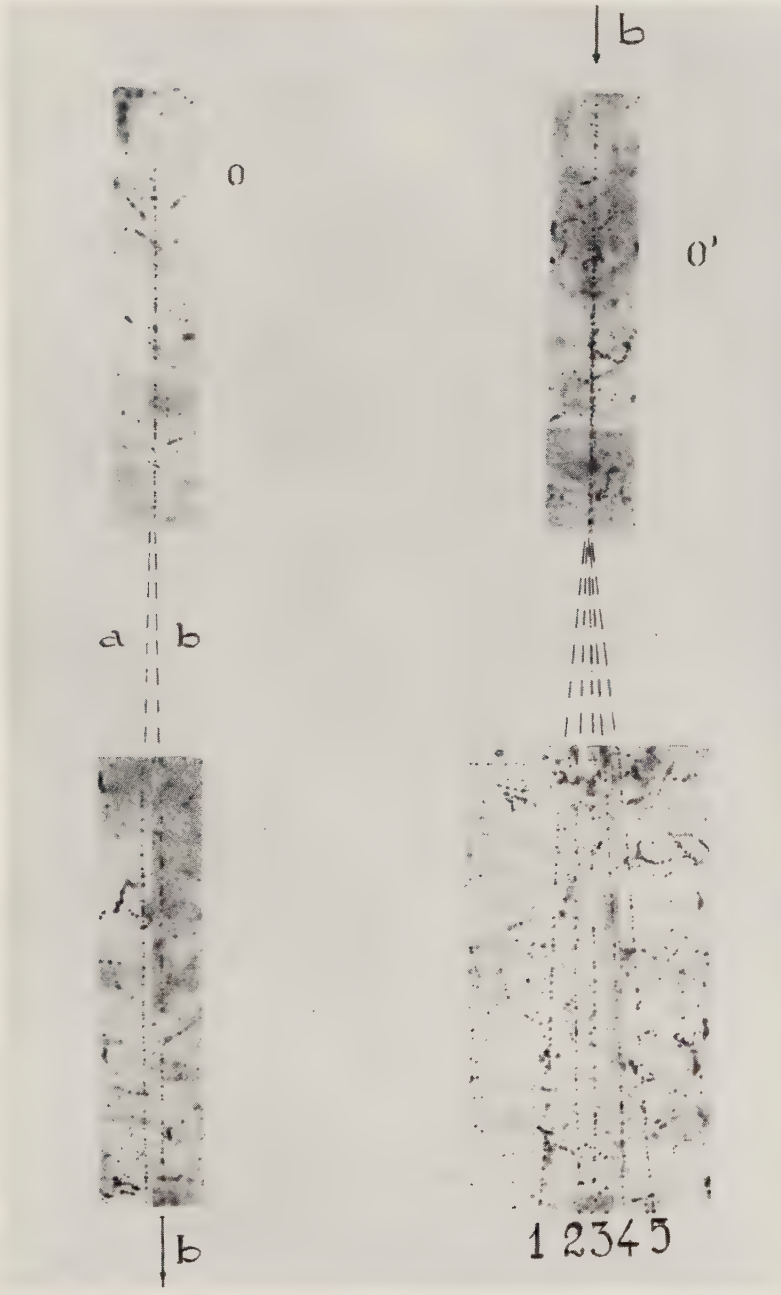

Fig. 1.

$= 5.3 \pm 0.3$, one notices that in the first 30 μm after the origin the number of grains n_2 is somewhat smaller than n_3.

pendently; 3) distinct origin for two tridents (or pseudo-tridents) separated by a distance Δx.

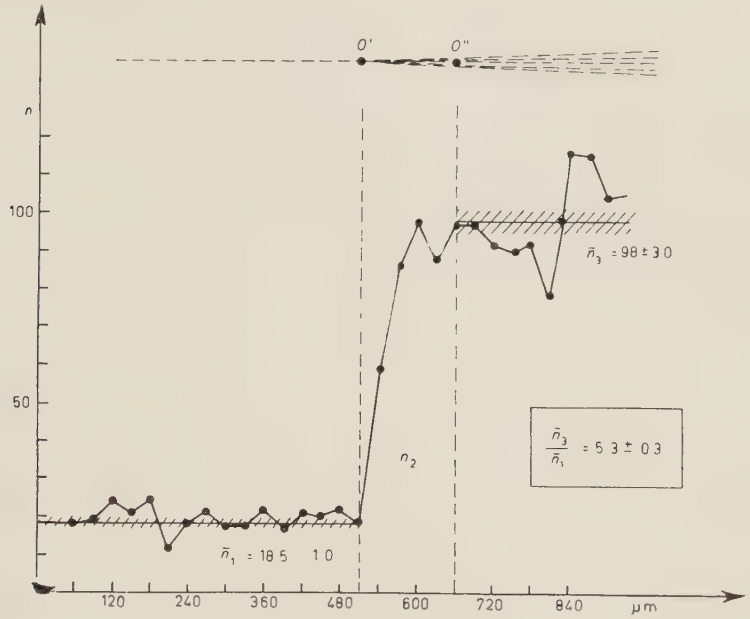

Fig. 2.

TABLE I.

Track		$p\beta$	i/i_0	angle
first pair	a	335 ± 50	1.03 ± 0.03	$\widehat{ab} = 1.2 \cdot 10^{-3}$ rad
	b	240 ± 40	1.03 ± 0.03	
double trident	1	—	1.0	$\widehat{1\text{-}5} = 1.6 \cdot 10^{-2}$ rad
	2	42 ± 4	0.97 ± 0.04	
	3	129 ± 28	1.00 ± 0.04	
	4	109 ± 16	1.05 ± 0.04	
	5	67 ± 6	0.96 ± 0.04	

This result, $n_2 < n_3$ may be due to: 1) statistical fluctuation; 2) overlapping of several minimum tracks. One can easily estimate (as control measurements on small aperture tridents have confirmed) that several overlapping minimum tracks give a total granulation smaller than that of the tracks inde-

Assuming the more conservative hypothesis, that is interpreting the values $n_2 < n_3$ as exclusively due to cause number 3, one can estimate $\Delta x \leqslant 30$ μm.

The cross-section ([4,5]) for pair pro-

([4]) H. J. BHABHA: *Proc. Roy. Soc.*, **152**, 559 (1935).

([5]) G. RACAH: *Nuovo Cimento*, **14**, 93 (1937).

duction by an electron of energy ε may be written in emulsion as

$$\sigma(\varepsilon) = 6.74 \cdot 10^{-28} \cdot \cdot \left\{ \ln^3 \varepsilon - 6.3 \ln^2 \varepsilon + 14 \ln \varepsilon - 15.4 \right\}.$$

For the primary energy $\varepsilon = 240$ MeV one gets $\sigma = 1.6 \cdot 10^{-26}$ cm^2 and for the secondaries $\sigma = 1.2 \cdot 10^{-26}$ cm^2. Since the number of atoms per cm^3 in G5 is $7.86 \cdot 10^{22}$ one gets a mean free path for trident production $\lambda = 790$ cm and 995 cm respectively.

The probability of the casual we seek comes out $\leqslant 10^{-5}$, and that for pseudotridents still lower ([6]).

It is to be noted that the cross-section for double pair production should be of the order $(1/137)^2$ of that for single production.

Dr. TEUCHER has pointed out to us that this type of event may be easily taken, during scanning, as a jet, thus possibly altering the statistics on these events.

([6]) R. WEILL, M. GAILLOUD and P. ROSSELET: *Helv. Phys. Acta*, **29**, 437 (1956); M. M. BLOCK, D. T. KING and W. WADA: *Phys. Rev.*, **96**, 1627 (1954); M. F. KAPLON and M. KOSHIBA: *Phys. Rev.*, **97**, 193 (1955); **100**, 327 (1955).

Renormalization of Axial Vector Coupling - II.

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(ricevuto il 17 Aprile 1958)

In a recent note (1) two axial vector currents were constructed that are approximately conserved, one for the case where the pion-baryon interaction is pseudoscalar and a scalar meson (*), σ, exists (case P), the other for the case where the pion-baryon interaction is pseudovector (case A). Both cases assumed that the bare meson masses were zero and that no meson-meson interactions existed. In this note we wish to discuss the extent to which these assumptions may be relaxed.

In case A the existence of the conserved current arose from the invariance of the theory under.

$$\boldsymbol{\pi} \to \boldsymbol{\pi} + \boldsymbol{\lambda} , \tag{1}$$

where $\boldsymbol{\lambda}$ is a constant. For invariance under (1) to hold it is essential that the pion bare mass should be zero. To maintain this invariance meson-meson interaction terms would have to consist solely of derivative couplings. However invariance under (1) will also have the unpleasant consequence that the *real* meson mass is zero also and so we reject this case altogether.

In case P let us first neglect the bare baryon mass. It is easy to see that meson-meson terms of the form

$$\lambda^{(n)}(\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \sigma\cdot\sigma)^n \tag{2}$$

maintain the conservation law since the law arises from the invariance of the theory under the set of transformations discussed by Schwinger (2) and (2) is also invariant under these transformations. The case with $n=1$ corresponds to an equal bare mass for $\boldsymbol{\pi}$ and σ. For the case $n=2$, $\lambda^{(2)}$ would be dimensionless.

It is possible to reintroduce the baryon bare mass by using the similarity between the bare mass term and the baryon-σ coupling (2). A theory consistent with the conservation law is obtained by replacing σ everywhere by $(\sigma+m_0/g)$. However some curious terms are produced such as

$$\lambda^{(2)}m_0^3\sigma/g^3 . \tag{3}$$

The bare mass terms presumably produce effects of the order of m_0/m and may plausibly be neglected since in case P we also neglect the medium-strong K-meson interactions.

(1) J. C. Polkinghorne: *Nuovo Cimento*, **8**, 179 (1958).

(*) The most direct way in which σ would be expected to manifest its existence is an $I=\frac{1}{2}$ resonance in pion-nucleon scattering. This could be the explanation of the well-known second resonance.

(2) J. Schwinger: *Ann. Phys.*, **2**, 407 (1957).

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Green's Functions in Meson Theories.

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(ricevuto il 23 Aprile 1958)

In some recent papers of Hiida [1,2] it is stated that the results obtained in reference [3] for the asymptotic Green's functions and vertex part do not agree with calculations based on perturbation theory. Hiida, however, does not present the calculations but simply cites the final result in the first order in g_0^2. All further arguments of the author are based on this result. A simple check of his calculations by employing standard formulae [4] indicates that an error was made by the author in his computations of the vertex part. The correct expression for neutral scalar (pseudoscalar) theory is

$$\Gamma(p^2) \equiv 1 - \frac{g_0^2}{4\pi} \ln \frac{\lambda^2}{-p^2},$$

which differs by the factor $-\frac{1}{2}$ before the logarithm from formula (2.3) in ref. [2] (the author mentions that his result differs from that in ref. [3] by a constant factor but does not attach any importance to this circumstance). Diversity of the coefficients in formulae Γ and S_F implies that in meson theories no identity holds which is similar to the Ward identity in electrodynamics and this is not consistent with what is asserted in ref. [1,2-5]. The source of the error in ref. [5] is that on differentiating S_F with respect to mass, account was not taken of the fact that during renormalization the expression

$$M(m) + (p - m) \frac{\partial M}{\partial p}\Big|_{p=m},$$

which is also mass dependent, should be subtracted from the mass operator.

In conclusion it may be mentioned that the formulae in ref. [3] were checked by straightforward perturbation theory calculations down to g_0^4 inclusively.

We conclude that Hiida is wrong in asserting that the results of ref. [3] are not consistent with perturbation theory.

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[1] K. Hiida: *Progr. Theor. Phys.* **17**, 520, 605 (1957).

[2] K. Hiida: *Nuovo Cimento*, **5**, 1094 (1957).

[3] A. A. Abrikosov, A. D. Galanin and I. M. Halatnikov: *Dokl.*, **97**, 793 (1954); A. D. Galanin, B. L. Ioffe and I. Ya Pomerančuk: *Žur. Eksp. Teor Fiz.*, **29**, 51 (1955); I. Ya. Pomerančuk, V. V. Čudakov and K. A. Ter-Martirosian: *Phys. Rev.*, **103**, 784 (1956).

[4] R. P. Feynman: *Phys. Rev.*, **76**, 769 (1949).

[5] K. Hiida and S. Machida: *Progr. Theor. Phys.*, **13**, 219 (1955).

Electromagnetic Properties of π-Meson.

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(ricevuto il 14 Maggio 1958)

The total charge-current operator taken between physical nucleon states with momentum-energy p, p', is given by [1],

$$\langle p' | J_\mu(0) | p \rangle = \bar{u}(p')[e\gamma_\mu F_1(q^2) + \mu\sigma_{\mu\nu}q^\nu F_2(q^2)]u(p) .$$

The form factors F_1, F_2, are real invariant functions of $q^2=(p-p')^2$ obeying the normalization prescriptions : $F_1(0)=F_2(0)=1$ for the proton and $F_1(0)=0$, $F_2(0)=1$ for the neutron; e and μ being the electric charge and the static anomalous magnetic moment of the corresponding nucleon.

The form factors have been calculated according to various models and have been experimentally determined by experiments on electron-nucleon scattering [2,3].

Reciprocally the total electromagnetic charge-current operator taken between physical meson states,

$$\langle p' | J_\mu(x) | p \rangle$$

will be affected by the meson-nucleon interaction, and can be determined again with pion-electron scattering experiments.

From translational invariance:

$$\langle p' | J_\mu(x) | p \rangle = \exp[i(p'-p)x] \langle p' | J_\mu(0) | p \rangle$$

and from gauge invariance:

$$(p'-p)_\mu \langle p' | J_\mu(0) | p \rangle = 0$$

identically, which requires

$$\langle p' | J_\mu(0) | p \rangle = e(p+p')_\mu f(p'^2, p^2, (p'-p^2)) , \tag{1}$$

f being an invariant function of p'^2, p^2, $(p-p')^2$.

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[1] L. L. Foldy: *Phys. Rev.*, **87**, 688 (1952); G. Salzman: *Phys. Rev.*, **99**, 973 (1955).

[2] D. R. Yennie, M. M. Levy and D. G. Ravenhall: *Rev. Mod. Phys.*, **29**, 144 (1957).

[3] R. Hofstadter, R. W. McAllister: *Phys. Rev.*, **98**, 217 (1955); R. W. McAllister and R. Hofstadter: *Phys. Rev.*, **102**, 851 (1956).

By comparison between the matrix element (1) and its transpose, it follows that

$$f(p'^2,\ p^2,\ (p'-p)^2) = f^*(p^2,\ p'^2,\ (p-p')^2)$$

such that for physical meson states $(p'^2=p^2=\mu^2)$, f is a real function of $(p'-p)^2=q^2$.

The $f(q^2)$ function is connected to the mean square radius of the e.m. charge-current distribution of the π-meson, $\langle r^2\rangle$, according to

$$f(q^2) = 1 - \frac{q^2}{6}\langle r^2\rangle + \dots .$$

The main difference between the nucleon « dimension » and the meson « dimension » is, apart from the two functions appearing in the nucleon current in contrast to the single function appearing in the pion current, in the fact that two different mechanisms give contributions to the nucleon charge radius, in contrast to just one in the case of the pion. In the nucleon case the actual radius is the sum of the « meson cloud radius », which arises from the direct interaction of the electromagnetic field with the virtual charged pion emitted by the nucleon, and of the « nucleon core radius » which is connected with the direct interaction of the nucleon with the electromagnetic field, the former oscillating in the fluctuation of the π vacuum field. If we assume a Yukawa meson-nucleon interaction, the meson dimension is simply the mean square radius of the pairs distributions weighted by the probability of absorption of the meson by a nucleon of the vacuum.

An estimation of the order of magnitude of this distance can be drawn from the uncertainty principle.

If the π-meson is at rest, the nucleon and the antinucleon will move in opposite direction with an average kinetic energy $\mu c^2/2$.

To this energy there corresponds a momentum $c\sqrt{\mu(M+(\mu/4))}$, which corresponds to a spreading of the wave function of the two fermions of the order of magnitude

$$\langle r\rangle = \frac{h}{c\sqrt{\mu(M+(\mu/4))}} \sim 5\cdot 10^{-14}\ \text{cm}\,.$$

Assuming a pseudoscalar meson-nucleon interaction and by keeping only terms in the first order contribution of the electromagnetic charge, it follows that [4]

$$\langle p'|J_\mu(0)|p\rangle = \left(\varphi_{p'}\Big|\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int \mathrm{d}t_1 \dots \mathrm{d}t_n P[\underline{H}_I(t_1)\,\underline{H}_I(t_2)\dots\underline{H}_I(t_n)\,\underline{J}_\mu(0)]\varphi_p\right)$$

where $J_\mu(0)$ is the interaction representation charge-current of meson plus nucleon field with neglect of the $A_\mu A_\mu \varphi\varphi^*$ term, $\varphi_{p'}$, φ_p are the meson bare states and $\underline{H}_I(t)$ is the meson-nucleon interaction energy in the interaction representation supplemented by the meson and nucleon self-energy and meson-meson interaction counter-terms.

(4) F. E. Low: *Phys. Rev.*, **97**, 1391 (1954).

By introduction of an external electromagnetic field,

$$A_i(x) = \delta_{i\mu}\delta^4(x)$$

we can rewrite

$$\langle p' | J_\mu(0) | p \rangle = \left(\varphi_{p'}, \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}t\, \mathrm{d}t_1 \ldots \mathrm{d}t_n\, P[H_I(t_1), H_I(t_2), H_I^{\mathrm{ext}}(t)] \varphi_p \right)$$

and we are left with the Feynman amplitude of meson scattering by the $A_i(x)$ field.

The renormalized contribution to the second order in the meson-nucleon interaction is

$$(2) \qquad -eG^2 \left[\int \mathrm{d}^4 r\, \mathrm{Tr} \left(\gamma_\mu \frac{1}{\not p + \not p' - M} \gamma_5 \frac{1}{\not r - M} \gamma_5 \frac{1}{\not p + \not r - M} \right) - \right.$$

$$\left. - \int \mathrm{d}^4 r\, \mathrm{Tr} \left(\gamma_\mu \frac{1}{\not p + \not p' - M} \gamma_5 \frac{1}{\not p - M} \gamma_5 \frac{1}{\not p + \not r - M} \right)_{p \to p'} \right] \qquad \not p \equiv p_\mu \gamma_\mu ;$$

where p and p' are the ingoing and outgoing meson momenta.

It is known that some caution is necessary in handling divergent electromagnetic charge-current in order not to lose the gauge invariance of the physical meaningful quantities.

After taking the trace in (2) one obtains an expression symmetric with respect to p and p', and one must pay attention to keep this symmetric property in the succeeding calculations, because it assures the gauge invariance of the results.

By neglecting terms of the order μ^2/M^2 and keeping only terms in q^2 one gets the results

$$\langle p' | J_\mu(0) | p \rangle = e(p + p')_\mu \left(1 - \frac{3}{20} \frac{G^2}{(2\pi)^2} \frac{q^2}{M^2} \right),$$

where G is the unrationalized, renormalized pseudoscalar coupling constant. By choosing $G = 15.5$, which fits the P-wave in (n-π) scattering, one gets $\langle r^2 \rangle^{\frac{1}{2}} = 4.9 \cdot 10^{-4}$ cm; but the intermediate states which we have taken into account in (2) are the same as those which are responsible for the S-wave (n-π) scattering and $\pi^0 \to 2\gamma$ decay, which needs a coupling constant approximately one tenth smaller.

Corresponding the e.m. radius becomes smaller.

In meson-nucleon scattering the energy necessary in the center of mass system for exploring the pion structure is consequently of the order of magnitude of the nucleon mass.

We notice that in the fusion theory of Fermi-Yang [5] the meson dimension has been estimated to be of the order of magnitude of the nucleon Compton wave-length.

From charge conjugation invariance considerations it follows that the electron-π^0 scattering vanishes.

As in electron-nucleon scattering experiments, the experimental detection of a form factor $f(q^2)$ is a proof of an e.m. structure of the π only if we assume that the conventional quantum electrodynamics does not break down at high energy, and

[5] E. Fermi and C. N. Yang: *Phys. Rev.*, **76**, 1739 (1949).

only electron-electron scattering experiments at high energy or Compton scattering on mesons can decide between the two alternatives.

By supposing that the charged pions are reciprocal charge conjugate spinless particles and assuming gauge invariance, from the validity of the CPT theorem one can state the equality of the form factors of π^+ and π^-.

In fact the electromagnetic charge-current operator transforms as

$$(CPT)^{-1} J_\mu(0)\, CPT = J_\mu^*(0) = -J_\mu(0)$$

consequently

$$\langle \pi^+, p \,|\, J_\mu(0) \,|\, \pi^+, p' \rangle = -\langle \pi^+, p \,|\, (CPT)^{-1} J_\mu(0)\, CPT \,|\, \pi^+, \mathrm{p}' \rangle = -\langle \pi^-, \mathrm{p}' \,|\, J_\mu(0) \,|\, \pi^-, \mathrm{p} \rangle ,$$

where in the last step we have exploited one consequence of the CPT theorem, which states that the energy spectrum is the same for particle and antiparticle ([6]). But from the gauge invariance:

$$\langle \pi^+, \mathrm{p} \,|\, J_\mu(0) \,|\, \pi^+, \mathrm{p}' \rangle , \qquad \langle \pi^-, \mathrm{p}' \,|\, J_\mu(0) \pi^-, \mathrm{p} \rangle ,$$

are symmetric for interchange of p with p' such that:

$$\langle \pi^+, p \,|\, J_\mu(0) \,|\, \pi^+, p' \rangle = -\langle \pi^-, p \,|\, J_\mu(0) \,|\, \pi^-, p' \rangle$$

and the form factor is the same for π^+ and π^-.

By imposing charge coniugation invariance, one would have arrived at the same result, but this condition is only sufficient and not necessary to establish this result.

([6]) T. D. LEE: *Conservation Laws in Weak Interactions* (Columbia University).

LIBRI RICEVUTI E RECENSIONI

F. SEVERI, *Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica*, Edizioni Cremonesi, Roma 1957, pag. 463 + IV.

Il presente volume costituisce il seguito di un noto precedente trattato dello stesso autore dal titolo *Serie, sistemi d'equivalenza e corrispondenze algebriche sulle varietà algebriche*, ed aggiunge un nuovo notevole contributo all'opera — veramente poderosa — del grande matematico italiano FRANCESCO SEVERI, nel campo della geometria algebrica.

Il trattato — che ha avuto origine dai corsi di lezioni tenute dall'autore all'Istituto Nazionale di Alta Matematica fin dal 1942 — concerne vasti e profondi argomenti di geometria algebrica, a cui qui non potremo che accennare fugacemente.

Più precisamente esso comprende la teoria dei sistemi continui di curve e sottovarietà algebriche contenute in una varietà algebrica, nonchè la teoria degli integrali semplici appartenenti ad una superficie e ad una varietà. Ad esso seguirà un terzo volume in cui sarà sviluppato il seguito della teoria degli integrali semplici e multipli appartenenti ad una superficie e ad una varietà algebrica.

Il volume, presentato in una eccellente veste tipografica dall'Editore Cremonese, è diviso in sei capitoli.

Il cap. I consta di due paragrafi. Il § 1 è dedicato alla teoria dei sistemi algebrici di curve piane. Nel § 2, dopo una interessante digressione sui sistemi algebrici di varietà, viene iniziata l'indagine sulle proprietà dei sistemi di curve sopra una superficie. Tale studio viene poi approfondito nel successivo capitolo.

Il cap. III è diviso in quattro paragrafi, nel primo dei quali l'autore esamina il sistema aggiunto di una curva generica di un sistema e ne dimostra la regolarità. Il § 2 è dedicato allo studio della varietà di Picard associata ad una superficie. Nel § 3 viene dato un nuovo interessante invariante birazionale delle superficie. Il § 4 chiude il cap. III con utili notizie storiche e bibliografiche e con profonde riflessioni critiche relative a quanto precedentemente svolto.

L'autore poi, dopo aver esposto nel cap. IV delle premesse di carattere topologico relative alle riemanniane delle superficie e varietà algebriche, svolge nel cap. V, in maniera profonda e in pari tempo limpidissima, la sua teoria della base sulle superficie e varietà algebriche, teoria che, come è noto, costituisce una delle maggiori glorie della geometria algebrica italiana. Il capitolo è diviso in cinque paragrafi, l'ultimo essendo dedicato a notizie e riflessioni storico-critiche sull'argomento. Nel § 1 l'autore tratta l'equivalenza algebrica sopra una varietà, mentre il § 2 è dedicato all'equivalenza razionale. Viene poi esaminato nel § 3 la teoria della base delle curve tracciate sopra una superficie, indi nel § 4, tale teoria viene estesa alle varietà algebriche.

Il cap. VI, che insieme al precedente costituisce il caposaldo di tutta l'opera, è dedicato alla teoria degli integrali semplici sopra una superficie. Esso è diviso in cinque paragrafi.

L'autore, dopo aver fatto nel § 1 delle premesse fondamentali sulla teoria degli integrali di forme differenziali, passa a trattare nel § 2 gli integrali semplici di 1ª, 2ª, 3ª specie sopra una superficie e fa uno studio preliminare di essi. Nel § 3 rivolge poi l'indagine agli integrali semplici di prima specie, mentre nel § 4 studia i cicli di una superficie in relazione con i cicli lineari di una sua curva. Infine nel § 5 l'autore approfondisce lo studio sugli integrali semplici di seconda specie di una superficie.

Concludendo, questo trattato — come già il precedente — per l'importanza e profondità degli argomenti trattati e la limpidezza con cui sono esposti, risulta di estrema utilità non solo per gli studiosi della geometria algebrica classica e moderna ma anche per un qualsiasi cultore delle scienze esatte, ed è destinato a rimanere un classico della letteratura matematica.

GIUSEPPE TALLINI

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